Math 4317, Assignment 4A

§1.3 Vector Spaces

If V and W be finite dimensional vector spaces over a field F, then V and W are **isomorphic** if there is a surjective linear transformation $T : V \to W$.

1. Given finite dimensional vector spaces V and W and a linear tranformation $T : V \to W$, show that $V/\operatorname{Ker}(T)$ and $T(V)$ are isomorphic and

$$\dim V = \dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T).$$

   Recall that the set $\operatorname{Im}(T)$ is also denoted $T(V)$.

2. Gunning §1.3 Group I Problem 4

§2.1 Normed Vector Spaces

3. Gunning §2.1 Group I Problem 2
4. Gunning §2.1 Group I Problem 3
5. Gunning §2.1 Group I Problem 4
6. Gunning §2.1 Group II Problem 5
7. Gunning §2.1 Group II Problem 7

§2.2 Metric Spaces: Less Structure than a Normed Vector Space

An **metric space** is a set X together with a function $d : X \times X \to [0, \infty)$ satisfying the following properties:

(i positive definite) $d(x, y) = 0$ if and only if $x = y$.
(ii symmetric) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The function $d : X \times X \to [0, \infty)$ is called a **distance function** or sometimes a **metric**.

Note that a metric space is **not** required to be a vector space, but may be just a set with no particular algebraic structure. Recall that a norm was also positive definite and so was an inner product. However, all three of these properties with the same names have different formulations. If we wish to distinguish them one from another, we may use the names:

positive definite property of a distance function (or metric)
positive definite property of a norm
positive definite property of an inner product

Similarly, we have

the triangle inequality for a distance function
(or the metric triangle inquality)
and
the triangle iequality for a norm.

There is also

metric symmetry
and
the symmetry of an inner product.

8. Show that $d(x, y) = \|x - y\|$ defines a distance function on any normed vector space. This is called the **norm induced metric**. Thus every normed space is a metric space.
9. Gunning §2.2 Group I Problem 1
10. Gunning §2.2 Group I Problem 2
11. Gunning §2.2 Group I Problem 4
12. Gunning §2.2 Group I Problem 6

Uniform Convergence

(We will use this in our study of monotone functions.)

Let $E \subset \mathbb{R}$ be a fixed set. If $u_j : E \to \mathbb{R}$ is a real valued function for each $j = 1, 2, 3, \ldots$ and $u : E \to \mathbb{R}$ is also a real valued function, we say the sequence

$$\{u_j\}_{j=1}^{\infty} \qquad \textbf{converges to } u \textbf{ pointwise}$$

if for each $x_0 \in E$ the following condition holds:

For any $\epsilon > 0$, there is some N such that

$$j > N \qquad \Longrightarrow \qquad |u_j(x_0) - u(x_0)| < \epsilon.$$

We say

$$\{u_j\}_{j=1}^{\infty} \qquad \textbf{converges uniformly to } u$$

if the following condition holds:

For any $\epsilon > 0$, there is some N such that

$$j > N \qquad \Longrightarrow \qquad |u_j(x) - u(x)| < \epsilon \qquad \text{for every } x \in E.$$

13. (a) Give an example of a sequence of real valued functions $u_j : \mathbb{R} \to \mathbb{R}$ that converges pointwise but not uniformly.
    (b) Give an example of a sequence of real valued functions $u_j \in C^0[0, 1]$ which converges pointwise but not uniformly.

14. (a) Give an example of a sequence of real valued functions $u_j \in C^0[0, 1]$ which converges pointwise to a discontinuous function.
    (b) Show that the uniform limit of a sequence of continuous real valued functions is continuous. More precisely, show that if $u_j : E \to \mathbb{R}$ is continuous at $x_0 \in E$ and $\{u_j\}_{j=1}^{\infty}$ converges uniformly to a function $u : E \to \mathbb{R}$, then u is continuous at x_0.

Monotone Functions

15. (a) Let $\{x_1, x_2, \ldots, x_n\}$ be a finite set of distinct real numbers. Let $\{y_1, y_2, \ldots, y_n\} \subset \mathbb{R}$, and let $\{a_1, a_2, \ldots, a_n\} \subset (0, \infty)$. Show $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x) = \sum_{j=1}^{n} u_j(x) \qquad \text{where} \qquad u_j(x) = \begin{cases} y_j, & x < x_j \\ y_j + a_j, & x \geq x_j \end{cases}$$

    is a monotone non-decreasing function with discontinuities precisely at $x_1, x_2, \ldots, x_n$.
    (b) Let

$$\{x_j\}_{j=1}^{\infty}$$

    be a sequence of distinct real numbers. Show there exists a monotone non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ with discontinuities precisely at $x_1, x_2, x_3, \ldots$.