Math 4317, Assignment 4B

§1.3 Vector Spaces

1. Given finite dimensional vector spaces V and W and a linear transformation $T : V \to W$, show T is continuous.

§2.1 Numbers and Arithmetic

2. If $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{1}{n}(a_1 + a_2 + \cdots + a_n). \tag{1}$$

The number $(a_1 + a_2 + \cdots + a_n)/n$ is called the **average** or **arithmetic mean**. The number on the left $(a_1 a_2 \cdots a_n)^{1/n}$ is called the **geometric mean**. Note that the geometric mean of a and b is $\sqrt{ab}$.

(a) Prove the arithmetic/geometric mean inequality (1).

(b) Show the (length of the) **altitude to the hypotenuse** of a right triangle is the geometric mean of the (lengths of the) segments into which it divides the hypotenuse.

(c) Illustrate the arithmetic/geometric mean inequality in $n = 2$, i.e., in $\mathbb{R}^2$, using right triangles.

§2.1 Normed Vector Spaces

3. Draw the unit ball in $\mathbb{R}^2$ in the ℓ^p "norm"

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{2} |x_j|^p \right)^{1/p}$$

where $\mathbf{x} = (x_1, x_2)$ and $p \in (0, \infty]$.

4. Gunning §2.1 Group II Problem 8

5. Gunning §2.1 Group II Problem 9

§2.2 Metric Spaces

Let X be a metric space with distance function d. The following definitions and problems are among the most important in analysis:

A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ **converges** to $z \in X$ if the following holds:

For any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$j > N \qquad \Longrightarrow \qquad d(x_j, z) < \epsilon.$$

In this case, we write

$$x_j \to z \ \ (\text{as } j \to \infty) \qquad \text{or} \qquad \lim_{j \nearrow \infty} x_j = z,$$

and z is called the **limit of the sequence**.

6. Show that the limit of a sequence in a metric space is unique. That is, if $x_j \to z$ and $x_j \to \tilde{z}$, then $z = \tilde{z}$.

A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ is called **Cauchy** if the following holds:

For any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$j, k > N \qquad \Longrightarrow \qquad d(x_j, x_k) < \epsilon.$$

7. If $x_j \to z$ in X, then $\{x_j\}_{j=1}^{\infty}$ is Cauchy.

8. Prove that any Cauchy sequence in a metric space is **bounded** in the sense that there is some $z \in \mathbb{R}$ and some $r > 0$ such that $d(x_j, z) < r$, i.e.,

$$\{x_j\}_{j=1}^{\infty} \subset B_r(z) = \{x \in X : d(x, z) < r\}.$$

Note: The element z is not the limit here.

Notice that the limit z of a sequence $\{x_j\}_{j=1}^{\infty}$ (if there is one) is not mentioned in the definition of a Cauchy sequence. The Cauchy condition may be thought of as providing a tool/means to think about (and prove things about) sequences without having to say anything about the limit of the sequence.

Definition: A metric space X is called **complete** (or metrically complete) if every Cauchy sequence converges (in X).

9. Show that the unit interval $(0, 1)$ with metric $d(x, y) = |x - y|$ is a metric space which is not complete and $\mathbb{R}$ with the same metric is (metrically) complete. Here you'll need to use Dedekind completeness and the assumption that the real numbers exist.

10. Gunning §2.2 Group II Problem 10

11. Gunning §2.2 Group II Problem 11

§2.3 Topology

A **topological space** is a set X together with a collection $\mathcal{T}$ of subsets of X satisfying

(i) $\phi, X \in \mathcal{T}$.

(ii) If $U_\alpha \in \mathcal{T}$ for $\alpha \in \Gamma$, then

$$\cup_{\alpha\in\Gamma} \in \mathcal{T}.$$

(iii) If $U_1, U_2, \ldots, U_k \in \mathcal{T}$, then

$$\cap_{j=1}^k U_j \in \mathcal{T}.$$

We say:

1. $\mathcal{T}$ is a topology (or collection of **open sets**) for X.
2. $\mathcal{T}$ is **closed under arbitrary unions**.
3. $\mathcal{T}$ is **closed under finite intersections**.

A set $A \subset X$ is **closed** if the complement of A is open.

The **metric topology** on a metric space X with distance function d is defined as follows: Given $r > 0$ and $x_0 \in X$, the set

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

is called the **open ball** with radius r and center x_0. The metric topology is then

$$\mathcal{T}_d = \{U \subset X : \text{ for any } x_0 \in U, \text{ there is some open ball } B_r(x_0) \subset U\}.$$

12. (a) Show that in a general topological space any intersection of closed sets is closed. Given any set $A \subset X$ where X is a topological space, the particular intersection

$$\cap\{C : C \text{ is closed and } C \supset A\}$$

is called the **closure** of A and is denoted by $\overline{A}$.

(b) Show that the following are equivalent for a set $A \subset X$ of a topological space X:

(i) $A \cap U \neq \phi$ for every open set U.

(ii) $\overline{A} = X$.

If these (equivalent) conditions hold, we say A is **dense** in X.

(c) Show the metric topology $\mathcal{T}_d$ is a topology.

(d) (Now that we have the metric topology on a metric space X and know what it means for a set to be open) show an open ball $B_r(x_0)$ is open.

§2.2 The Real Numbers

Let $X = \mathbb{Q}$ be the rational numbers with distance $d(x, y) = |x - y|$ and let R be the collection of all Cauchy sequences of rational numbers.

Now, we have to be a little bit careful here. We defined a distance function to be a function $d : X \times X \to [0, \infty)$ with certain properties. Now, if we want to construct $\mathbb{R}$ from $\mathbb{Q}$, then we don't have $\mathbb{R}$ yet, and so, we don't have $[0, \infty)$ as an interval in $\mathbb{R}$ yet either (!).

13. Let $d : X \times X \to [0, \infty)$ where $X = \mathbb{Q}$ and

$$[0, \infty) = \{x \in \mathbb{Q} : x \geq 0\}.$$

Give a **generalized definition** of a metric space which allows this possibility. Can you define Cauchy sequences, convergence, completeness, and boundedness for this kind of metric space? Is it still true that every Cauchy sequence is bounded?

14. Consider the following:

$$\{x_j\}_{j=1}^{\infty} + \{\xi_j\}_{j=}^{\infty} = \{x_j + \xi_j\}_{j=1}^{\infty}, \qquad \{x_j\}_{j=1}^{\infty}\{\xi_j\}_{j=}^{\infty} = \{x_j \xi_j\}_{j=1}^{\infty},$$

and

$$c\{x_j\}_{j=1}^{\infty} = \{cx_j\}_{j=1}^{\infty}.$$

What kind of algebraic structure(s) do these put/allow on R? (Be sure to consider if these formulas give well-defined operations on R.)

15. Notice that it doesn't make sense to talk about convergence for a lot of sequences in $\mathbb{Q}$, but it does make sense to talk about **convergence to zero**. Define What it means for a sequence in $\mathbb{Q}$ to converge to zero, and show that

$$\{x_j\}_{j=1}^{\infty} \sim \{\xi_j\}_{j=1}^{\infty} \qquad \Longleftrightarrow \qquad \lim_{j \to \infty} |\xi_j - x_j| = 0$$

defines an equivalence relation on R.