Math 4317, Assignment 5A

§1.3 Vector Spaces

1. Any finite dimensional vector space V over a field F is isomorphic (as a vector space) to F^n.

§2.1 Normed Vector Spaces

2. Show that the space $C^0[a,b]$ of all **real valued continuous functions** on an interval $[a,b]$ is a complete normed vector space under the norm

$$\|f\| = \sup\{|f(x)| : x \in [a,b]\}.$$

A (metrically) complete normed vector space is called a **Banach space**.

§2.1 Inner Product Spaces

Here we will **assume** there is a continuous decreasing bijective function $\cos : [0,\pi] \to [-1,1]$ with graph as indicated in Figure 1.

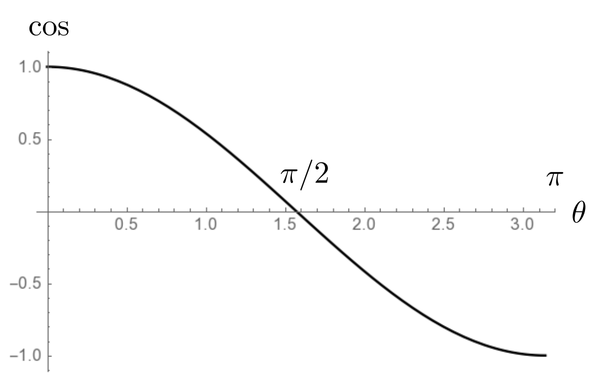

Figure 1: The Cosine

At this point we have not defined the cosine function, and we have not defined the real number π, but you are familiar with the idea that such a function exists. Note: The fact that we (i.e., you) have not rigorously defined the cosine function and we do not have a solid definition of the number π should give you great motivation to get through Chapter 3 of Gunning. In the mean time, let's use what you've heard about cosine.

3. Explain how the values of $\cos\theta$ indicated in Figure 1 relate to the coordinates of (and the angles associated to) the points in

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1,\ y \geq 0\}.$$

4. Show that any decreasing function $f : E \to \mathbb{R}$ (where $E \subset \mathbb{R}$) has a well-defined inverse $f^{-1} : f(E) \to E$. Draw the graph of the inverse cosine.

5. In **any inner product space** V, define a function $\theta : V\backslash\{0\} \times V\backslash\{0\} \to [0, \pi]$ by

$$\theta(\mathbf{v}, \mathbf{w}) = \cos^{-1}\left(\frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right).$$

   (a) Show θ is well-defined (assuming inverse cosine is well-defined).

   (b) Show θ is continuous where the metric on $V\backslash\{\mathbf{0}\} \times V\backslash\{\mathbf{0}\}$ is induced from the norm on $V \times V$ given by

$$\|(\mathbf{v}, \mathbf{w})\|_\times = \sqrt{\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2}.$$

   (c) Show θ cannot be extended continuously to $V \times V\backslash\{\mathbf{0}\}$.

6. If V is an inner product space, show

$$\langle\!\langle (\mathbf{v}, \mathbf{w}), (\mathbf{x}, \mathbf{y}) \rangle\!\rangle = \langle \mathbf{v}, \mathbf{x}\rangle + \langle \mathbf{w}, \mathbf{y}\rangle$$

   is an inner product on the Cartesian product $V \times V$. What is the norm induced by $\langle\!\langle \cdot\, , \,\cdot \rangle\!\rangle$?

§2.3 Topology

Let X be a topological space. Define the **interior** of a set $A \subset X$ by

$$\operatorname{int} A = \bigcup_{U \subset A,\, U \text{ open}} U.$$

The interior is also sometimes denoted by A°. Define the **closure** of A by

$$\overline{A} = \bigcap_{C \text{ closed},\, C \supset A} C.$$

7. Let $A \subset X$. The set A is open if and only if

$$\operatorname{int} A = A.$$

   The set A is closed if and only if

$$\overline{A} = A.$$

The **boundary** of a set $A \subset X$ is defined by

$$\partial A = \overline{A} \cap \overline{A^c} \qquad \text{where} \qquad A^c = X\backslash A.$$

8. $\operatorname{int} A = A\backslash\partial A$ and $\overline{A} = A \cup \partial A$.

9. $$\partial A = \{x \in X : \text{every open set } U \text{ with } x \in U \text{ has } U \cap A \neq \phi \text{ and } U \cap A^c \neq \phi\}.$$

10. A is both closed and open if and only if $\partial A = \phi$.

Uncountable Sums (of non-negative real numbers)

I remember reading somewhere about uncountable sums of non-negative numbers. I thought I had read about this topic in some book by Walter Rudin, but when I look in all my books by Walter Rudin, I can't seem to find the topic. In some sense these are kind of exotic objects, and they can be mostly avoided and mostly lived without. But sometimes they can be convenient, and we will use them in our study of monotone functions below. I think I remember the definition and here it is:

Given any indexing set Γ and any collection of non-negative numbers x_α indexed by Γ, we define

$$\sum_{\alpha\in\Gamma} x_\alpha = \sup\left\{\sum_{\alpha\in I} x_\alpha : I \subset \Gamma,\ \#I < \infty\right\}.$$

This will be the main definition we will use.

11. Show that $\sum_{\alpha\in\Gamma} x_\alpha$ is always a well-defined extended real number which is (possibly) finite only if $\Gamma\backslash\Gamma_0$ is countable where

$$\Gamma_0 = \{\alpha \in \Gamma : x_\alpha = 0\},$$

i.e., $x_\alpha = 0$ except for a countable collection of indices.

Another interesting definition of the sum of arbitrary collections of numbers is the following: Given $\{x_\alpha\}_{\alpha\in\Gamma} \subset \mathbb{R}$, we write

$$\sum_{\alpha\in\Gamma} x_\alpha = z \in \mathbb{R}$$

if given any $\epsilon > 0$, there is some finite subcollection $I_0 \subset \Gamma$ such that

$$\left|\sum_{\alpha\in I} x_\alpha - z\right| < \epsilon \qquad \text{whenver } I \text{ is a finite set with } I_0 \subset I \subset \Gamma.$$

Obviously a variant can be formulated for sequences in any metric space, and the requirement can be strengthened by requiring the inequality for all countable sets of indices I instead of just finite ones. I don't know too much about these notions.

Extension of u_- and u_+: Given and interval $I \subset \mathbb{R}$ and a non-decreasing function $u : I \to \mathbb{R}$, recall that if $x \in I$ and there is some open interval $(a, b) \subset I$ with $x \in (a, b)$, then

$$u_-(x) = \sup\{u(\xi) : \xi < x\} \qquad \text{and} \qquad u_+(x) = \sup\{u(\xi) : \xi > x\}$$

are well-defined. We can also write these as

$$u_-(x) = \lim_{\xi \nearrow x} u(\xi) = \lim_{\xi \to x^-} u(\xi) \qquad \text{and} \qquad u_+(x) = \lim_{\xi \searrow x} u(\xi) = \lim_{\xi \to x^+} u(\xi).$$

If $x \in I$ and there is no open interval (a, b) with $x \in (a, b) \subset I$, then either $x = a = \inf I = \min I$ or $x = b = \sup I = \max I$. In these two cases (if one of them arises) we set

$$\begin{aligned} u_-(a) &= u(a) \qquad \text{if } a = \min I, \text{ and} \\ u_+(b) &= u(b) \qquad \text{if } b = \max I. \end{aligned}$$

Definition (semicontinuity) Given a topological space X and a real valued function $f : X \to \mathbb{R}$, we say f is **lower semicontinuous** if $\{x : f(x) > v\}$ is open for every $v \in \mathbb{R}$. A function $f : X \to \mathbb{R}$ is **upper semicontinuous** if $\{x : f(x) < v\}$ is open for every $v \in \mathbb{R}$.

12. Given a non-decreasing function $u : I \to \mathbb{R}$, the function $u_- : I \to \mathbb{R}$ non-decreasing and lower semicontinuous and the function $u_+ : I \to \mathbb{R}$ is non-decreasing and upper semicontinuous.

Given an interval $I \subset \mathbb{R}$ with $a \in I$ and two numbers $b, c \in [0, \infty)$ with $b \le c$ and $\max b, c \in (0, \infty)$ the (non-decreasing) function $w : I \to \mathbb{R}$ given by

$$w(x) = \begin{cases} 0, & x < a \\ b, & x = a \\ c, & x > a \end{cases}$$

is called a **normalized step function with jump of height c at $a \in I$.**

A function $w : I \to \mathbb{R}$ is called a **jump function** if w can be written as a sum

$$w(x) = \sum_{j \in \Gamma} w_j(x) \tag{1}$$

for some normalized step functions w_j with jumps in I indexed by $j \in \Gamma \subset \mathbb{N}$.

13. Let $w_j : I \to \mathbb{R}$ be a normalized step function, and let $w : I \to \mathbb{R}$ be a jump function.
    (a) Under what conditions is w_j lower semicontinuous? Upper semicontinuous? Both lower and upper semicontinuous?
    (b) Show that the representation (1) of a jump function w is never unique.

(c) Let $u : I \to \mathbb{R}$ be a non-decreasing function and $a \in \operatorname{int} I$. Assume the restrictions of u to the intervals $I \cap (-\infty, a)$ and $I \cap (a, \infty)$ are each continuous but u is not continuous. Consider the functions $w_j : I \to \mathbb{R}$ and $w : I \to \mathbb{R}$ given by

$$w_j(x) = \begin{cases} 0, & x \leq a \\ u(a) - u_-(a), & x = a \\ u_+(a) - u_-(a), & x > a \end{cases}$$

and

$$w(x) = u(x) - u_-(x) + [u_+(\xi) - u_-(\xi)] \qquad \text{for } \xi \in I \text{ fixed.}$$

(i) Show w_j is a normalized jump function.

(ii) Calculate the value of $w(x) - w_j(x)$ when $\xi < x < a$.

(d) Let $v : I \to \mathbb{R}$ by $v(x) = u(x) - w_j(x)$.

(i) Show v is non-decreasing.

(ii) Find $v_-(a)$ and $v_+(a)$.

(iii) Show $v \in C^0(I)$.

14. Let $u : I \to \mathbb{R}$ be a non-decreeasing function. Let $w : I \to \mathbb{R}$ by

$$w(x) = u(x) - u_-(x) + \sum_{\xi < x} [u_+(\xi) - u_-(\xi)].$$

(a) Show w is a well-defined (finite valued) jump function.

(b) Show $u(x) = v(x) + w(x)$ where $v \in C^0(I)$.

15. Let $J(I, x_0)$ denote the collection of all (non-decreasing) jump functions $w : I \to \mathbb{R}$ with $w(x_0) = 0$. Let $ND(I)$ denote the collection of all non-decreasing functions $u : I \to \mathbb{R}$.

(a) Show that for each $u \in ND(I)$, there exist functions $w \in J(I, x_0)$ and $v \in C^0(I)$ with

$$u = v + w.$$

(b) Show that if $u \in ND(I)$ is written as $u = v + w$ for some $w \in J(I, x_0)$ and $v \in C^0(I)$, then v and w are unique and v is non-decreasing.

(c) Is $ND(I)$ a vector space? Is $J(I, x_0)$ a vector space?