Math 4317, Assignment 6A

This is not an official assignment of Math 4317 in the Spring semester of 2020, but it contains some extra problems that seemed to be interesting but either not central enough to the subject matter of the course to cover or that were just left out because there was not enough time to cover them. This additional assignment may be of interest to give the student some idea of "what he was missing" from the course.

§2.2 Metric and Normed Spaces

1. Gunning §2.2 Group II Problem 13

The previous problem gives an isometry $\phi : X \hookrightarrow V$ of any arbitrary metric space X into a normed vector space V.

We know the metric d on X may not come from a norm even if X is a vector space. See the discussion of on page 73 of Gunning concerning the **discrete metric**

$$d_s(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$$

2. Take a vector space X, like $X = \mathbb{R}$ or $X = \mathbb{R}^n$, with a metric, like d_s that does not come from a norm. Why does
$$\|p\|_X = \|\phi(p)\|_V,$$
where V is the normed vector space from the previous problem and $\phi : X \hookrightarrow V$ is the isomporphism from the previous problem, **not** define a norm on V? What goes wrong?

§2.3 Topology

A topological space X is called **separable** if there exists a countable set

$$\{x_j\}_{j=1}^{\infty} \subset X$$

which is **dense** in X, that is, $\overline{A} = X$.

3. Show any interval of $\mathbb{R}$ is separable.

§2.4 Baire Category Theorem

4. Prove any non-singleton interval $I \subset \mathbb{R}$ is uncountable.

Postmortem: Actually, I did include the Baire category theorem on the final assignment.

The most prominent/serious omission in the course is the material of Gunning §3.3 (of Chapter 3) concerning series and series of functions in particular. We did touch on series of functions, but this section gives the main results on power series. Using power series is the ususal way to define the standard transcendental functions like the exponential function, natural logarithm, and the trigonometric functions. Covering this material is important, and I view it as unfortunate that we didn't do that. With the trigonometric functions and the material on series, one can consider Fourier series, which is also technically on the syllabus for this course, but I view this as less central (and certainly less standard) for a first semester in analysis.

On the bright side, a student who mastered the material we did cover in the course should be very well equipped to read and master all of Chapter 3 of Gunning which also contains a great deal more than material on series including differentiability for functions of several variables. This same comment applies, of course, to the preponderance of material on Assignment 5 and the final assignment (and to some extent the material on Assignment 4 and even Assignment 3). I think it can hardly be said that most students have mastered all the material on Assignments 1 through 3. Still, almost every student mastered something, even if only one or two problems from assignments 1 and 2. And at whatever point you find yourself, the material I have given you (along with Gunning's text) is relatively well-organized and puts you in a position to continue your progress until you master elementary analysis.

At this point, I should mention that, in addition to Gunning's text, I used a great deal of material from the introductory chapter of Giovanni Leoni's excellent text *A First Course in Sobolev Spaces*. He dedicated the text to his advisor Jim Serrin, who was an amazing mathematician and expositor, and Leoni has produced a text that I think comes up to the very high standards of Serrin and honors the memory of his mathematics well. It was my (ambitious) intention to include the details of the proof that every monotone function on an interval is differentiable on a subset having the same "measure" as the interval. Two things can be said about this: In the spirit of what I have written above, we definitely didn't come close to achieving this goal, but the interested student should be in a relatively good position to pick up Leoni's book and read about the necessary topics.

Were I to teach this course again and/or reorganize the Assignments, it might be a reasonable idea to cut essentially all the algebra. I had originally felt that about the "right" amount of material could be encapsulated in 5 assignments with 30 problems each plus a final assignment of 30 problems. With each of these assignments being covered in about three weeks, this would work, I think. We were simply not able to keep up that pace. In the end, I think we essentially covered in detail material equivalent to approximately four of the assignments, so something more than half the material we "should" have covered. Hopefully what we have lacked in quantity we have made up for in quality. There are certainly a half dozen students who participated in MATH 4317 Spring semester 2020 who are in a good position to easily complete all the material we "should" have covered. It is my hope that each and every one of my students will (be inspired to) do this.

§3.1 Limits and Continuity

5. Gunning §3.1 Group I Problem 1

6. Gunning §3.1 Group I Problem 2

7. Gunning §3.1 Group I Problem 4
8. Gunning §3.1 Group II Problem 6
9. Gunning §3.1 Group II Problem 7 This is a poorly worded problem...of course, there is, more or less, only one way in which the problem can be read that makes sense. What Gunning is trying to say is:

   Let E be a closed subset of $\mathbb{R}$. Show that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

   $$f(x) = 0 \quad \text{for every } x \in E \qquad \text{and} \qquad f(x) \neq 0 \quad \text{for every } x \in \mathbb{R}\backslash E.$$

10. Gunning §3.1 Group II Problem 9

§3.2 Differentiability

We have introduced derivatives and differentiability for functions of one real variable using the formulation of Leoni. Gunning uses a more standard (and less general) approach in which one considers $u : U \to \mathbb{R}$ where U is an open set. In fact, he allows the open set U to be a subset of $\mathbb{R}^n$, so that functions of several variables are considered, and he allows vector valued functions as well. In the special case when U is an open subset of $\mathbb{R}$ and $u : U \to \mathbb{R}$, the definitions of Leoni and Gunning are equivalent—though they do not necessarily look equivalent.

I do not like the approach of Gunning in which the derivative (matrix) of a transformation and the linear mapping/approximation function are identified using the same notation "A." This goes back to Gunning's discussion of linear algebra in §1.3 and specifically to pages 39 through, say, 45. He introduces linear transformations, or a linear transformation, using the symbol "T" as in $T : \mathbb{R}^n \to \mathbb{R}^m$, and then formally suggests using the same symbol "A" for the matrix associated with T and for the transformation T as well. This convention is used, for example, in the statement of Theorem 1.15 on page 45 of Gunning.

Because I don't like this convention, I will briefly give the definition of differentiability using the notation I prefer. Please compare the following to the beginning of §3.2.

Given an open set $U \subset \mathbb{R}^n$ and a point $\mathbf{p} \in U$, a function $\mathbf{f} : U \to \mathbb{R}^m$ is said to be **differentiable** at $\mathbf{x}_0$ if there is a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ for which

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{x}_0 + \mathbf{v}) - \mathbf{f}(\mathbf{x}_0) - L(\mathbf{v})}{|\mathbf{v}|} = \mathbf{0}.$$

NOtice that this limit is being taken in the metric space $\mathbb{R}^m$ and involves function values for a function evaluated on $\mathbb{R}^n$. Thus, the quantity $|\mathbf{v}|$ is the Euclidean norm of $\mathbf{v}$ in $\mathbb{R}^n$. In particular, the same condition could be expressed in terms of real limits as follows:

$$\lim_{|\mathbf{v}|\to 0} \left| \frac{\mathbf{f}(\mathbf{x}_0 + \mathbf{v}) - \mathbf{f}(\mathbf{x}_0) - L(\mathbf{v})}{|\mathbf{v}|} \right| = 0.$$

If $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in U$, then the **derivative** of $\mathbf{f}$ at $\mathbf{x}_0$ is the matrix of the linear transformation L which we denote by

$$D\mathbf{f}(\mathbf{x}_0).$$

Recall that the **matrix associated with a linear transformation** $L : \mathbb{R}^n \to \mathbb{R}^m$ is the unique $m \times n$ matrix A for which $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Thus, in terms of the derivative $A = D\mathbf{f}(\mathbf{x}_0)$, if it exists, the condition of differentiability may be expressed as

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{x}_0 + \mathbf{v}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{v}}{|\mathbf{v}|} = \mathbf{0}.$$

Derivatives Without Differentiability

We have now defined the derivative of a differentiable function as the matrix of a certain linear transormation which approximates the values of the function near a point $\mathbf{x}_0$. It will be recalled that Leoni (Final Assignment) defined the derivative as the limit of the difference quotient at a point x_0 in the extended real numbers (and then defined differentiability in terms of the derivative).

11. If U is an open subset of $\mathbb{R}^n$ and $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable, then $\mathbf{f} = (f_1, f_2, \dots, f_m)$ for some real valued functions $f_j : U \to \mathbb{R}$, $j = 1, 2, \dots, m$, and the derivative (matrix)

$$A = (a_{ij}) = D\mathbf{f}(\mathbf{x}_0)$$

has entries satisfying

$$a_{ij} = \lim_{h\to 0} \frac{f_i(\mathbf{x}_0 + h\mathbf{e}_j) - f_i(\mathbf{x}_0)}{h} \tag{1}$$

where $\mathbf{e}_j$ is the vector in $\mathbb{R}^n$ with 1 in the j-th entry and zeros elsewhere. The entry a_{ij} of $D\mathbf{f}(\mathbf{x}_0)$ and, more generally, the limit of the difference quotient in (1) when it exists is denoted by all of the following:

$$\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0) = D_j f_i(\mathbf{x}_0) = D^{\mathbf{e}_j} f_i(\mathbf{x}_0)$$

and is also called the **partial derivative** of the compoenent function f_i in the direction $\mathbf{e}_j$, or with respect to x_j. Gunning also uses $\partial_j f_i$ and $\mathbf{f}'(\mathbf{x}_0)$ for the matrix $D\mathbf{f}(\mathbf{x}_0)$. These notations are not some common.

12. If U is an open subset of $\mathbb{R}^1$ and $\mathbf{r} : U \to \mathbb{R}^m$, then differentiability at $\mathbf{x}_0$ and the existence of a derivative defined by a difference quotient are equivalent. (Prove it.)

13. If U is an open subset of $\mathbb{R}^n$ and $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0$, then $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

14. Find an example of a function $\mathbf{f} : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ such that all the partial derivatives of $\mathbf{f}$ exist at some point $\mathbf{x}_0 \in U$, but $\mathbf{f}$ is **not** differentiable at $\mathbf{x}_0$ and $\mathbf{f}$ is also **not** continuous at $\mathbf{x}_0$.

Lebesgue Outer Measure

Leoni uses this concept in his discussion of monotone functions. Given **any** set $E \subset \mathbb{R}^1$, the **Lebesgue outer measure** of E is defined to be

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty}(b_j - a_j) : E \subset \bigcup_{j=1}^{\infty}(a_j, b_j) \right\}. \tag{2}$$

15. (a) Show that $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ with values given by (2) is a well-defined extended real valued function.

    (b) Show thaf if $E_1 \subset E_2$, then $\mu^*(E_1) \leq \mu^*(E_2)$. We say that μ^* is a **monotone set function**.

    (c) Show thaf if $E_1, E_2, E_3, \ldots$ is a sequence of subsets of $\mathbb{R}$, then

    $$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(E_j).$$

    We say that μ^* is **subadditive**.

    (d) If $I_1, I_2, I_3, \ldots$ is a countable sequence of disjoint intervals with endpoints a_j and b_j satisfying $a_j \leq b_j$, then

    $$\mu^*\left(\bigcup_{j=1}^{\infty} I_j\right) \leq \sum_{j=1}^{\infty}(b_j - a_j).$$

    (e) If $E \subset \mathbb{R}$, then

    $$\mu^*(E) = \inf\{\mu^*(U) : U \text{ is open in } \mathbb{R} \text{ with } E \subset U\}.$$

    (f) If I is an interval with endpoints a and b satisfying $a \leq b$, then

    $$\mu^*(I) = b - a.$$

    In particular, the Lebesgue outer measure of any singleton (our countable union of single points) is zero.