Math 4317, Final Assignment Part A

§1.3 Vector Spaces

1. Prove Gunning's Theorem 1.5: If A is an $m \times n$ matrix with entries in a field F, then there are square invertible matrices $T_{m\times m}$ and $U_{n\times n}$ such that
$$TAU = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$
where I_k is the $k \times k$ identity matrix and $\dim ker(A) = n - k$.

2. Gunning §1.3 Group I Problem 7

§2.4 Compact Sets

3. Prove any compact subset of a Hausdorff space is closed.

4. Prove any compact subset of a metric space is bounded.

5. (Heine-Borel) Prove any closed and bounded subset of $\mathbb{R}^n$ is compact.

6. Let X be a complete metric space. If $E_1, E_2, E_3, \ldots$ is a sequence of closed and bounded sets satisfying the following
   1. $E_j \neq \phi$ for $j = 1, 2, 3, \ldots,$
   2. $E_j \supset E_{j+1}$, and
   3. $\lim_{j\to\infty} \operatorname{diam} E_j = 0,$

   then the intersection
$$\cap_{j=1}^{\infty} E_j \quad \text{contains precisely one point.}$$

7. (Baire) Let X be a (nonempty) complete metric space. If $G_1, G_2, G_3, \ldots$ is a sequence of dense open sets in X, i.e., each G_j is open and
$$\overline{G}_j = X,$$
then
$$\overline{\cap_{j=1}^{\infty} G_j} = X.$$
In particular, $\cap_{j=1}^{\infty} G_j \neq \phi$. Hint: Use the result of the previous problem.

   Incidentally, a countable intersection of open sets is called a "G-δ" set. A set is **nowhere dense** if its closure contains no open set. A countable union of nowhere dense sets is called (according to Baire) a **set of the first category**. A set is called (according to Baire) **a set of the second category** if it is not a set of the first category. Use this lovely inspired terminology of Baire to describe the result above.

8. Prove Gunning's Corollary 2.30: The set of rational numbers cannot be written as a countable intersection of open sets.

9. Prove Gunning's Corollary 2:32: $\mathbb{R}$ is uncountable.

§3.1 Limits and Continuity

We have seen $\epsilon - \delta$ continuity for real valued functions defined on subsets of $\mathbb{R}$ in Assignment 3B. The definition there generalizes immediately to functions on metric spaces: If X and Y are metric spaces and $f : X \to Y$ is a function, then f is **continuous** at $x_0 \in X$, if for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d_X(x, x_0) < \delta \qquad \Longrightarrow \qquad d_Y(f(x), f(x_0)) < \epsilon.$$

We have also seen various notions of limits of sequences of numbers. These definitions contain the basic ideas for a rigorous definition of **convergence at finite points** for a function. Again, let X and Y be metric spaces and consider a function $f : X \to Y$ be a function from one metric space X to another metric space Y. We say $f(x)$ **converges to a limit** $y_0 \in Y$ if for any $\epsilon > 0$, there exists some δ such that

$$x \in B_\epsilon(x_0)\backslash\{x_0\} \qquad \Longrightarrow \qquad f(x) \in B_\delta(y_0).$$

It is often convenient to have a more general version of this definition: If $A \subset X$ and $f : A \to Y$, we say $f(x)$ **converges to a limit** $y_0 \in Y$ if for any $\epsilon > 0$, there exists some δ such that

$$x \in [B_\epsilon(x_0)\backslash\{x_0\}] \cap A \qquad \Longrightarrow \qquad f(x) \in B_\delta(y_0).$$

Notice the second version does not require f to be defined at the point $x_0 \in X$. In either of these cases, we write

$$\lim_{x \to x_0} f(x) = y_0.$$

10. Show $f : X \to Y$ is continuous at $x_0 \in \text{clus}(X)$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Does this result still hold if $x_0 \in X \backslash \text{clus}(X)$? N.b. Gunning Theorem 3.11.

11. Show that a function $f : X \to Y$ is continuous at every $x \in X$ (or simply **continuous on** X) if and only if $f^{-1}(V)$ is open for every open set V in Y.

If $f : X \to Y$ is a function, and $Y = \mathbb{R}$, we say f is a **real valued function**. The set of all continuous real valued functions on a metric space X is denoted $C^0(X)$.

12. If X is a metric space and $x_0 \in X$, then $f : X \to \mathbb{R}$ by $f(x) = d(x, x_0)$ is continuous on X, i.e., $f \in C^0(X)$.

13. If X is a normed space, then $f : X \to \mathbb{R}$ by $f(x) = \|x\|$ is continuous.

14. Show the continuous image of a connected set is connected.

15. Show the continuous image of a compact set is compact.

16. (extreme value theorem) If $f \in C^0(K)$ where K is a compact set, then

$$-\infty < \inf_{x \in K} f(x) \le \sup_{x \in K} f(x) < \infty$$

and there exist points x_m and x_M in K such that

$$f(x_m) = \inf_{x \in K} f(x) \qquad \text{and} \qquad f(x_M) = \sup_{x \in K} f(x).$$

This means, in particular, that

$$\min_{x \in K} f(x) \qquad \text{and} \qquad \max_{x \in K} f(x) \qquad \text{both exist.}$$

17. (Intermediate value theorem) If $f \in C^0[a,b]$ with $f(a) \neq f(b)$ and v is any real number (strictly) between $f(a)$ and $f(b)$, then there exists some $x_* \in (a,b)$ with $f(x_*) = v$.

18. (sequential limits) If X and Y are metric spaces with $x_0 \in \mathrm{clus}(X)$ and $f : X \to Y$, show

$$\lim_{x \to x_0} f(x) = y_0$$

if and only if for every sequence $\{x_j\}_{j=1}^{\infty} \subset X$ with $\lim_{j \to \infty} x_j = x_0$, there holds

$$\lim_{j \to \infty} f(x_j) = y_0.$$

Does this result still hold if $x_0 \in X \backslash \mathrm{clus}(X)$? N.b. Gunning Theorem 3.12.

Oscillation

The **oscillation** of a bounded real valued function $f : X \to \mathbb{R}$ defined on a metric space X over the set $A \subset X$ is defined by

$$\mathrm{osc}(f, A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x).$$

The **oscillation at a point** $x_0 \in X$ is defined by

$$\mathrm{osc}(f, x_0) = \lim_{r \to 0} \mathrm{osc}(f, B_r(x_0)).$$

19. Show the oscillation of a bounded function at a point is well-defined and that such a function is continuous at $x_0 \in X$ if and only if $\mathrm{osc}(f, x_0) = 0$.

20. (a) Show there exists a function which is discontinous at each rational number and continuous at each irrational number. Hint: Recall Assignment 4A Problem 15.
    (b) Show there does **not** exist a function which is discontinuous at each irrational number and continuous at each rational number.

Homeomorphism

Let X and Y be topological spaces. A function $f : X \to Y$ is a **homemorphism** if

1. f is a bijection.
2. f is continuous.
3. f^{-1} is continuous.

21. (a) Find an example of a function $f : X \to Y$ which is a bijection and is continuous but is **not a homeomorphism**, i.e., $f^{-1} : Y \to X$ is not continuous.

    (b) (Theorem 3.10 of Gunning) Show that a bijective continuous function $f : X \to Y$ where X is a compact Hausdorff space and Y is a Hausdorff space **is a homeomorphism**. Note/recall that the requirement that X and Y be Hausdorff is not very restrictive; the main hypothesis is that X is compact.

Uniform Continuity

Let X and Y be metric spaces. A function $f : X \to Y$ is **uniformly continuous** if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x_2, x_1) < \delta \qquad \Longrightarrow \qquad d_Y(f(x_2), f(x_1)) < \epsilon. \tag{1}$$

Notice that δ here does not depend on either point x_1 or x_2 individually. Of course, δ may be expected to depend on ϵ, and the assertion (1) depends on the closeness of the points x_1 and x_2 as stipulated.

22. (a) Show that if X and Y are normed vector spaces and $f : X \to Y$ is any continuous **linear** function, then f is uniformly continuous.

    (b) Can you formulate a notion of uniform continuity applicable to general topological spaces? That is, if X and Y are topological spaces (with no metric) and $f : X \to Y$ is continuous, is there any natural notion of uniform continuity for f?

    (c) Show that if X and Y are metric spaces, X is compact, and $f : X \to Y$ is continuous, then f is uniformly continuous.

    (d) Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : (0, 1) \to \mathbb{R}$ which are **not uniformly continuous**.

Uniformly Cauchy Sequence of Functions

A sequence of functions $f_j : X \to Y$ for $j = 1, 2, 3, \ldots$ where X is a topological space and Y is a metric space is said to be **uniformly Cauchy** if for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$j, k > N \qquad \Longrightarrow \qquad d_Y(f_j(x), f_k(x)) < \epsilon \quad \text{for all } x \in X.$$

Remember the definition of **uniform convergence** of a sequence of real valued functions defined on a subset of $\mathbb{R}$ from Assignment 4A. That definition extends naturally to this context: A sequence of functions $f_j : X \to Y$ for $j = 1, 2, 3, \ldots$ where X is a topological space and Y is a metric space is said to be **uniformly convergent** to a function $f : X \to Y$ if for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$j > N \qquad \Longrightarrow \qquad d_Y(f_j(x), f(x)) < \epsilon \quad \text{for all } x \in X.$$

23. (a) Show that any uniformly convergent sequence of functions is uniformly Cauchy.
    (b) Show that any uniformly Cauchy sequence of functions $f_j : X \to Y$ for $j = 1, 2, 3, \ldots$ **with values in a complete metric space** Y is uniformly convergent to some function $f : X \to Y$.

Differentiability

If $E \subset \mathbb{R}$ and $x_0 \in E \cap \text{clus}(E)$, a function $u : E \to \mathbb{R}$ is said to have **an extended real valued derivative** $u'(x_0)$ at x_0 if

$$u'(x_0) = \lim_{\substack{x \in E \\ x \to x_0}} \frac{u(x) - u(x_0)}{x - x_0}$$

exists as an extended real number in $[-\infty, \infty]$. Given a function $u : E \to \mathbb{R}$ with extended real valued derivative $u'(x_0)$, we say u is **differentiable** at x_0 if $u'(x_0) \in \mathbb{R}$.

The quantity

$$\frac{u(x) - u(x_0)}{x - x_0}$$

is called the **difference quotient** at x_0. The difference quotient may also be expressed as

$$\frac{u(x_0 + h) - u(x_0)}{h}$$

where $h = x - x_0 \neq 0$.

24. (a) Show that if $a_0, a_1, \ldots, a_k \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ by

    $$p(x) = \sum_{j=0}^{k} a_j x^j,$$

    then p is differentiable (and find the formula for $p'(x)$).
    (b) Show that if $u : E \to \mathbb{R}$ is differentiable at $x_0 \in E$, then u is continuous at x_0.
    (c) Show $f : \mathbb{R} \to \mathbb{R}$ by

    $$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

    is nowhere differentiable.

Weierstrass' Nowhere Differentiable Function

In this construction we will use the sine and cosine functions. Recall that we have not yet rigorously defined the cosine function or the number π, but we are all moderately familiar with the fact (mentioned in Assignment 5A) that there is a well-defined monotone decreasing bijective function $\cos : [0, \pi] \to [-1, 1]$. As mentioned in Assignment 5A, this function is continuous. Furthermore, the cosine function extends to the entire real line as a **differentiable** periodic function $\cos : \mathbb{R} \to [-1, 1]$ satisfying

$$\cos(\theta + 2\pi) = \cos\theta \qquad \text{and} \qquad \cos(\theta + \pi) = -\cos\theta \quad \text{for every } \theta \in \mathbb{R}$$

as indicated in Figure 1.

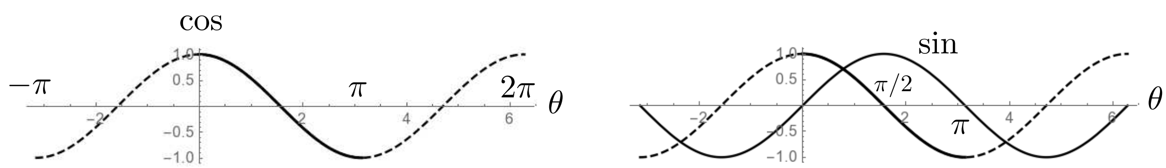

Figure 1: The cosine and the sine

The extension is no longer bijective or invertible on this new domain, but it is surjective. Now, we will assume also that the cosine function is differentiable and has a continuous derivative which we denote by

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

The function $\sin : \mathbb{R} \to [-1, 1]$ may also be assumed to have the familiar monotonicity, periodicity and differentiability properties:

$$\sin : [-\pi/2, \pi/2] \to [-1, 1] \qquad \text{is continuous surjective and increasing,}$$

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \sin(\theta + \pi) = -\sin\theta, \text{ and} \qquad \frac{d}{d\theta}\sin\theta = \cos\theta \quad \text{for every } \theta \in \mathbb{R}.$$

We can (and should) prove all these properties rigorously. There are some things we can prove now.

25. Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to have a **local maximum** at $x_0 \in \mathbb{R}$. Show that if f is differentiable at a point of local maximum x_0, then $f'(x_0) = 0$. Conclude that $\sin k\pi = \cos(\pi/2 + k\pi) = 0$ for $k \in \mathbb{Z}$.

We will also use the following properties of the cosine function:

$$|\cos\theta_2 - \cos\theta_1| \le |\theta_2 - \theta_1| \qquad \text{for all } \theta_1, \theta_2 \in \mathbb{R},$$

$$\cos[\theta_1 + \theta_2] = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \qquad \text{for all } \theta_1, \theta_2 \in \mathbb{R},$$

and

$$2 + 3\pi < 15.$$

26. Consider the function $u : \mathbb{R} \to \mathbb{R}$ given by

$$u(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(15^j \pi x).$$

Show that u is well-defined and continuous. Hint(s): Remember from assignment 4A that a uniform limit of continuous functions is continuous.

27. Consider Weierstrass' continuous function $u : \mathbb{R} \to \mathbb{R}$ defined in the previous problem. Let $u_k : \mathbb{R} \to \mathbb{R}$ be the partial sum

$$u_k(x) = \sum_{j=0}^{k} \frac{1}{2^j} \cos(15^j \pi x).$$

Let $x_0 > 0$ be fixed.

(a) Show the difference quotient for u_k at x_0 satisfies

$$\left| \frac{u_k(x_0 + h) - u_k(x_0)}{h} \right| < \pi \, \frac{(15/2)^{k+1}}{15/2 - 1}.$$

(b) Write $15^{k+1} x = m_k + \delta_k$ where $m_k \in \mathbb{N}_0$ and $-1/2 < \delta_k \leq 1/2$, and consider the sequence

$$h_k = \frac{1 - \delta_k}{15^{k+1}}.$$

Show the difference quotient for $u - u_k$ at x_0 with increment h_k satisfies

$$\frac{(u - u_k)(x_0 + h_k) - (u - u_k)(x_0)}{h_k} = \frac{(-1)^{m_k+1}}{h_k} \sum_{j=k+1}^{\infty} \frac{1}{2^j} \left[1 + \cos(15^{j-k-1} \delta_k \pi)\right],$$

and

$$\left| \frac{(u - u_k)(x_0 + h_k) - u_k(x_0)}{h_k} \right| \geq \frac{1}{h_k} \frac{1}{2^{k+1}} \left[1 + \cos(\delta_k \pi)\right] \geq \frac{1}{h_k} \frac{1}{2^{k+1}}.$$

(c) Conclude that

$$\lim_{k \to \infty} \left| \frac{u(x_0 + h_k) - u(x_0)}{h_k} \right| = +\infty.$$

and hence, u is **not differentiable at any point** $x_0 \in \mathbb{R}$.

28. (Mean value theorem, Gunning Theorem 3.29) If f and g are continuous functions on the closed interval $[a, b]$ and f and g are both differentiable on the open interval (a, b), then there exists some $x_* \in (a, b)$ with

$$[f(b) - f(a)] g'(x_*) = [g(b) - g(a)] f'(x_*).$$

Hint: Apply the extreme value theorem to $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$.

29. (Mean value theorem, Gunning Theorem 3.29) If $f \in C^0[a, b]$ and f is differentiable on the open interval (a, b), then there exists some $x_* \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(x_*).$$

Monotone Functions (inverses)

30. Let $I = [a, b]$ be an interval in $\mathbb{R}$ and $u : I \to \mathbb{R}$ a non-decreasing left continuous function. Consider $v : J \to \mathbb{R}$ by
$$v(y) = \inf\{x \in I : u(x) \geq y\}.$$
Show that for all $x \in [a, b]$
$$u(x) = \inf\{y \in [u(a), u(b)] : v(y) \geq x\}.$$