Cantor-Bernstein Theorem
Solution of Exercise 4

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1 Introduction

The statement of Exercise 4 in the discussion of the Cantor-Bernstein theorem says the following:

Show that if our strategy for this proof is going to work, then we must have

$$A\backslash F \supset \bigcup_{k\in\mathbb{N}_0} (g\circ f)^k[g(B\backslash f(A))]. \tag{1}$$

In the Cantor-Bernstein theorem we are given injections $f : A \to B$ and $g : B \to A$. The suggested strategy is to define a bijection $h : A \to B$ having the form

$$h(x) = \begin{cases} f(x) & \text{for } x \in F, \\ g^{-1}(x) & \text{for } x \in A\backslash F \end{cases} \tag{2}$$

where F is the set appearing in this exercise, and we are trying to find that set. For this exercise, we need to assume h is a bijection determined by the given definition (2) and then prove the required set inclusion (1). It's also sort of important to note that, at this point in the discussion, we have not already taken $F = F_0$ in order to prove the Cantor-Bernstein theorem. Up to this point, the set F is just some unknown set for which the definition of h might give a bijection. In particular, there is no particular relation between F and the set later defined as F_0.[1]

[1] Technically, it was claimed prior to Exercise 4 that $F_0 \subset F$, so if we needed to use that we probably could, but we do not need that.

2 Solution

To show the set inclusion, let us start with a point x for which

$$x \in \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k [g(B \backslash f(A))], \tag{3}$$

This means

$$x = (g \circ f)^k \circ g(b) \quad \text{for some } k \in \mathbb{N}_0 \text{ and some } b \in B \backslash f(A).$$

Let us consider a sequence of mappings

$$\begin{array}{ccccccccccc} B & \to & A & \to & B & \to & A & \to & B & \to & \cdots \\ b & \mapsto & g(b) & \mapsto & f \circ g(b) & \mapsto & (g \circ f) \circ g(b) & \mapsto & f \circ (g \circ f) \circ g(b) & \mapsto & \cdots \end{array}$$

The even elements (second, fourth, sixth, and so on) in this sequence are all in A and each has the form

$$(g \circ f)^\ell \circ g(b) \quad \text{for } \ell = 0, 1, 2, \ldots.$$

The elements in B have the form

$$f \circ (g \circ f)^m \circ g(b) \quad \text{for } m = 0, 1, 2, \ldots$$

except for the first one, which is just b. Referring to these as the "A elements" and the "B elements," our first claim is that all the A elements are distict (i.e., different from each other). Similarly, all the B elements are distinct. To see the first claim, simply note that if

$$(g \circ f)^\ell \circ g(b) = (g \circ f)^{\ell + j} \circ g(b),$$

then

$$b = f \circ (g \circ f)^{j-1} \circ g(b) \in f(A). \tag{4}$$

And this is a contradiction because $b \notin f(A)$. For the B elements, we can consider the first one b separately, but that leads immediately to

$$b = f \circ (g \circ f)^m \circ g(b) \in f(A)$$

which is just as much a contradiction as (4). Some other pair of these B elements being equal looks like

$$f \circ (g \circ f)^m \circ g(b) = f \circ (g \circ f)^{m+j} \circ g(b),$$

which implies (4) as well.

Now we are going to attempt a kind of induction to show $x \in A \backslash F$. Let's first show the first A element $g(b)$ has $g(b) \in A \backslash F$. Proceeding by contradiction, assume $g(b) \in F$. Then we know $h \circ g(b) = f \circ g(b)$, and we know $f \circ g(b) \neq b$. To restate this so we can refer back to it later

$$h(g(b)) \neq g^{-1}(g(b)) = b. \tag{5}$$

On the other hand, h is surjective, so there is some $\xi \in A$ with $h(\xi) = b$. Notice that $\xi \notin A_0$ because this would give $h(\xi) = f(\xi) = b$ contradicting the fact that $b \notin f(A)$. Therefore, $\xi \in A \backslash F$ and

$$h(\xi) = g^{-1}(\xi) = b$$

which means $\xi = g(b)$. In view of (5) this is also a contradiction. We have shown $g(b) \in A \backslash F$.

To understand the inductive step, realize that $x = (g \circ f)^k \circ g(b)$ is one of the A elements in our sequence. So we assume, in general, that we have the previous A element in $A \backslash F$. That is,

$$\xi = (g \circ f)^{k-1} \circ g(b) \in A \backslash F.$$

The sequence at this point looks like

$$\begin{array}{ccccccccc} \cdots & \to & A & \to & B & \to & A & \to & \cdots \\ \cdots & \mapsto & (g \circ f)^{k-1} \circ g(b) & \mapsto & f \circ (g \circ f)^{k-1} \circ g(b) & \mapsto & (g \circ f)^k \circ g(b) & \mapsto & \cdots \end{array}$$

The element between ξ and x, namely

$$\beta = f \circ (g \circ f)^{k-1} \circ g(b) \in B$$

is such that $f(\xi) = \beta = g^{-1}(x)$. Since h is onto and only defined by the values of f and g^{-1}, one of these equalities (at least one of them) must define the bijection h. On the other hand, since h is injective, and $x \neq \xi$, we know at most one of them defines h.

If we assume $h(\xi) = f(\xi) = \beta$, then this means $\xi \in F$. But our inductive assumption is $\xi \in A \backslash F$, so this is not what happens. The alternative is that

$$h(x) = g^{-1}(x) = \beta,$$

and this means $x \in A \backslash F$ as we needed to show. $\square$