Cantor-Bernstein Theorem
Solution of Exercise 6

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January 2, 2020

Introduction

The statement of Exercise 6 in the discussion of the Cantor-Bernstein theorem is the following:

Consider

$$h_1(x) = \begin{cases} f(x) & \text{for } x \in A\backslash G_0, \\ g^{-1}(x) & \text{for } x \in G_0. \end{cases}$$

What happens if one tries to show h_1 is a bijection? What about in the explicit example?

Here, as we know, we have injections $f : A \to B$ and $g : B \to A$. Also, the set G_0 is defined by

$$G_0 = \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k [g(B\backslash f(A))],$$

and we know from Exercise 4 that if $x \in G_0$ then $h_1(x) = g^{-1}(x)$ is well-defined, and if h_1 is to be a bijection, then that is how $h_1(x)$ must be defined.

The "explicit example" is given by $f(n) = 2n$ and $g(m) = 2m+1$ where $A = B = \mathbb{N}_0$.

1 Solution Part A

As in the proof that h was a bijection, the function h_1 is clearly well-defined. This is because $G_0 \subset g(B)$.

We can consider three cases in which $h_1(a) = h_1(x)$ to see that h_1 is an injection:

CASE I: $a, x \in A \backslash G_0$.

In this case, $h(a) = f(a)$ and $h(x) = f(x)$. Since f is injective, we know $a = x$.

CASE II: $a \in G_0$ and $x \in A \backslash G_0$.

$a = (g \circ f)^k \circ g(b)$ for some $k \in \mathbb{N}_0$ and some $b \in B \backslash f(A)$. If $k = 0$, then $f(x) = b$ which contradicts $b \notin f(A)$. If $k > 0$, then

$$f(x) = f \circ (g \circ f)^{k-1} \circ g(b) \qquad \text{or} \qquad x = (g \circ f)^{k-1} \circ g(b) \in G_0.$$

This is also a contradiction.

CASE III: $a, x \in G_0$.

In this case, $g^{-1}(a) = g^{-1}(x)$, so $a = x$ simply by application of g to both sides.

Therefore, h_1 is injective.

When we try to show h_1 is surjective, there seems to be a problem. We start with $b \in B$, and then we can consider $g(b) \in A$. Of course, if $g(b) \in G_0$, then we have $a = g(b) \in A$ with $h_1(a) = g^{-1}(a) = b$. So that's okay. But if $g(b) \in A \backslash G_0$, then it is not immediately clear how to find some $x \in A$ for which $h_1(x) = f(x) = b$.

If we knew

$$g(b) = (g \circ f)^m(a) \in F_0 = \bigcup_{n \in \mathbb{N}_0} (g \circ f)^n (A \backslash g(B))$$

then we would be okay. If $m = 0$, we get $a = g(b)$ which contradicts $a \notin g(B)$, so we know $m > 0$, and this means

$$b = f \circ (g \circ f)^{m-1}(a)$$

and since $x = (g \circ f)^{m-1}(a) \in F_0 \subset A \backslash G_0$ by Exercise 5, then we have $b = f(x) = h_1(x)$. But we don't know $g(b)$ is in one of the sets F_0 and G_0. Maybe $g(b)$ is some point in A outside both these sets. So we're stuck for the moment.[1]

Solution Part B

We recall that the first set defining F_0 was $A \backslash g(B)$. In the explicit example, we have

$$g(B) = \{2m + 1 : m \in \mathbb{N}_0\} \qquad \text{(the positive odd integers)}$$

[1]Note that one way to view the basic problem here is that we haven't said anything about, and we don't know anything about, the set $A \backslash (F_0 \cup G_0)$. In order to get further, we need to figure out something about the elements in $A \backslash (F_0 \cup G_0)$ if there are any. This is addressed below, and it turns out something can be said about those elements.

and

$$A\backslash g(B) = \{2m : m \in \mathbb{N}_0\} \qquad \text{(the non-negative even integers).}$$

Then we had more interesting sets.

$$g \circ f[A\backslash g(B)] = \{8m + 1 : m \in \mathbb{N}_0\} = \{1, 9, 17, 25, \ldots\},$$

$$(g \circ f)^2[A\backslash g(B)] = \{32m + 5 : m \in \mathbb{N}_0\} = \{5, 37, 69, 101, \ldots\}.$$

These latter sets are all odds of course. They go along with the evens to make up F_0.

On the other side, G_0 starts with $g(B\backslash f(A))$ and is all composed of odd numbers:

$$g(B\backslash f(A)) = \{4n + 3 : n \in \mathbb{N}_0\} = \{3, 7, 11, 15, \ldots\},$$

$$g \circ f[g(B\backslash f(A))] = \{16n + 13 : n \in \mathbb{N}_0\} = \{13, 29, 45, 61, \ldots\},$$
$$(g \circ f)^2[g(B\backslash f(A))] = \{64n + 53 : n \in \mathbb{N}_0\} = \{53, 117, 245, 309, \ldots\}.$$

One may sort of suspect all odd numbers will show up in this process so that $A = \mathbb{N}_0 = F_0 \cup G_0$ in this explicit example. In fact, this is the case, but we basically need to think about a different proof of the Cantor-Bernstein theorem by Julius König to see it. König considers sequences alternating with values between A and B like we did in the solution to Exercise 4, but with two new ingredients. The first is that instead of just starting at a particular element $b \in B$ and applying g and then f alternatively, we also consider the possibility of starting at some element $a \in A$ and then applying f followed by g and so on. The other new ingredient, is that König considers going in the reverse direction as well. Say you have

$$b \mapsto g(b) \mapsto f \circ g(b) \mapsto \cdots$$

This will always continue to the right. But if $b \in f(A)$, then there is a unique continuation/extension to the left as well:

$$f^{-1}(b) \mapsto b \mapsto g(b) \mapsto f \circ g(b) \mapsto \cdots .$$

In our explicit example, the most important observation is that these sequences are all decreasing and bounded below (by 0). This means each such sequence must stop (on the left). There are precisely two ways such a sequence can stop. One way is that one ends up with an element of $B\backslash f(A)$, in which case the last element on the left is in B, the second element is in G_0, and every other element after that (the A elements) are all in G_0. Conversely, every element of G_0 is, by definition, one of the A elements in such a sequence.

The other possibility for stopping on the left is that you end up with an element of $A\backslash g(B)$. In this case, the A elements in this sequence are in F_0. And just like for the elements in G_0, this is precisely what it means to be in F_0.

To summarize, Königs construction reinterprets the elements of F_0 as those which fall in one of these bi-directional sequences which ends with an element in A. The elements in G_0 are those elements in A found in bi-directional sequences ending in B.

In our explicit example, all sequences end, and every integer is clearly in some sequence. Thus, $\mathbb{N}_0 = F_0 \cup G_0$ in our example. In particular, all the odds will be found in the sequence of sets indicated above. Also, the function $h_1 : \mathbb{N}_0 \to \mathbb{N}_0$ defined as in this exercise will be a bijection in our explicit example.

2 Solution Part C

Königs construction also clears up something else. The elements in A which are in $A\backslash(F_0 \cup G_0)$ are precisely those which do not end on the left. It is pretty easy to see that there can be such examples. Let's take $A = B = \mathbb{Z}$ and for clarity, let's denote the elements of A by a_j for $j \in \mathbb{Z}$ and the elements of B by b_j for $j \in \mathbb{Z}$. Then consider

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is odd or positive,} \\ n-1 & \text{if } n \text{ is even and nonpositive.} \end{cases}$$

and

$$g(m) = \begin{cases} m+1 & \text{if } m \text{ is even,} \\ m-1 & \text{if } m \text{ is odd.} \end{cases}$$

That is,

$$f : a_{2j+1} \mapsto b_{2j+2}, \qquad a_{|j|+1} \mapsto b_{|j|+2}, \qquad a_{-2|j|} \mapsto b_{-2|j|-1},$$

and

$$g : b_{2j+1} \mapsto a_{2j}, \qquad b_{2j} \mapsto a_{2j+1}.$$

Notice that $b_1 \in B\backslash f(A)$, so

$$b_1 \mapsto a_0 \mapsto b_{-1} \mapsto a_{-2} \mapsto b_{-3} \mapsto \cdots$$

is a sequence ending on the left in B. Thus, $a_0, a_{-2}, a_{-4}, \ldots \in G_0$.

On the other hand, a_1 satisfies

$$\cdots \mapsto b_{-2} \mapsto a_{-1} \mapsto b_0 \mapsto a_1 \mapsto \cdots$$

and this sequence does not end on the left. Therefore,

$$\ldots, a_{-3}, a_{-1}, a_1, a_3, \ldots \in A\backslash(F_0 \cup G_0).$$

So our proof above for h_1 would nominally be in trouble. It is also possible, of course, that one of these sequences that does not end on the left can repeat instead of containing infinitely many elements. This is the case, for example if $f(a) = b$ and $g(b) = a$. Then you can go back and forth alternating between a and b, but you still get $a \in A\backslash(F_0 \cup G_0)$.

Now, we can complete our proof. Recall that we started with $b \in B$. We considered the cases when $g(b) \in G_0$ and when $g(b) \in F_0 \subset A\backslash G_0$. In each of these cases, we found an element $x \in A$ for which $h_1(x) = b$. The final case is that in which $g(b)$ lies in the set $U = A\backslash(F_0 \cup G_0)$ consisting of all the elements in A belonging to unending sequences, i.e., sequences that do not end on the left. Notice that the B elements in these sequnces are also disjoint from any sequence generating elements in F_0 or G_0 because they are all also elements in unending sequences. Thus, we have defined $h_1(x) = f(x)$ for all these elements. In particular, proceeding to the left from $g(b) \in U \subset A$, we have

$$\cdots \mapsto x \mapsto b \mapsto g(b) \mapsto \cdots$$

for some $x \in A$ with $f(x) = b$. Since $x \notin G_0$ is a part of a sequence which is unending on the left, we have $h_1(x) = f(x) = b$, and h_1 is onto. $\square$

Epilogue

The sets

$$F_0 = \bigcup_{n \in \mathbb{N}_0} (g \circ f)^n (A\backslash g(B)) \qquad \text{and} \qquad G_0 = \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k [g(B\backslash f(A))]$$

in our explicit example seem to be quite interesting. Let us write

$$\Phi_j = (g \circ f)^j (A\backslash g(B)) \qquad \text{and} \qquad \Psi_k = (g \circ f)^k [g(B\backslash f(A))]$$

so that $F_0 = \cup\Phi_j$ and $G_0 = \cup\Psi_k$. We know that the "base sets" $A\backslash g(B)$ and $g(B\backslash f(A))$ consist of the evens and certain odds, namely $\{4n + 3 : n \in \mathbb{N}_0\}$, respectively. We know further that the sets Φ_j for $j \in \mathbb{N}$ and Ψ_k for $k \in \mathbb{N}_0$ contain all the positive odd integers.

Exercise 1 *Show that for $j \in \mathbb{N}$*

$$\Phi_j = \{\phi_{n,j} : n \in \mathbb{N}_0\} \qquad \text{where} \qquad \phi_{n,j} = 2^{2j+1}n + \sum_{\ell=0}^{j-1} 4^\ell.$$

Show that for $k \in \mathbb{N}_0$

$$\Psi_k = \{\psi_{n,k} : n \in \mathbb{N}_0\} \qquad \text{where} \qquad \psi_{n,k} = 2^{2k+2}n + 2 \cdot 4^k + \sum_{\ell=0}^{k} 4^\ell.$$

Show that the sets Φ_j, $j \in \mathbb{N}$ and Ψ_k, $k \in \mathbb{N}_0$ are disjoint sets of odd integers, so that the formulas given for $\phi_{n,j}$ and $\psi_{n,k}$ represent unique odd integers. Hint on the formulas: Induction. Hint on the last part: König's construction.

If the assertions in the previous exercise are correct, then we have represented the odd integers as the image of the disjoint union of the two integer lattices $L = \mathbb{N}_0 \times \mathbb{N}$ and $M = \mathbb{N}_n \times \mathbb{N}_0$.

A standard proof of the countability of the (non-negative) rational numbers involves a mapping from the lattice $L = \mathbb{N}_0 \times \mathbb{N}$ (with the lattice point (n, k) representing the rational number n/k and ignoring reduction to lowest terms, so 1/2 and 2/4 are considered "different") to the integers. One possibility is

$$\nu(n, k) = \frac{(n + k - 1)(n + k)}{2} + k.$$

This map starts with $(0, 1) \mapsto 1$. Then you move to the next "anti-diagonal" starting with $(1, 1) \mapsto 2$ and $(0, 2) \mapsto 3$. The next anti-diagonal is

$$(2, 1) \mapsto 4, \qquad (1, 2) \mapsto 5, \qquad (0, 3) \mapsto 6,$$

and so on. This is a very simple pattern associating a natural number to each element of the lattice.

It is interesting to observe how the formulas for ψ and ϕ above associate distinct odd naturals numbers to each node in $L \cup M$. (Can you see a pattern?)