Matrix Center

John McCuan

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Here is my attempt to clearly explain that if $AB = BA$ for all $n \times n$ matrices B, then the $n \times n$ matrix A must be diagonal and have the form aI for some scalar a where I is the $n \times n$ identity matrix:

Consider the matrix E_{ij} where the element in the i, j position is equal to 1 and all other entries are 0. The product AE_{ij} is a matrix with all zeros except (possibly) in the j-th column, and the j-th column has entries $a_{1i}, a_{2i}, \ldots, a_{ni}$:

$$AE_{ij} = \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \\ & & & \downarrow & & & \end{array} \\ \begin{pmatrix} 0 & & 0 & a_{1i} & 0 & & 0 \\ 0 & & 0 & a_{2i} & 0 & & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & & 0 & a_{ni} & 0 & & 0 \end{pmatrix} \end{array}.$$

Note that the ij entry in this matrix is a_{ii}. The matrix $E_{ij}A$ on the other hand, is a matrix with all zeros except (possibly) in the i-th row. The i-th row, moreover contains entries $a_{j1}, a_{j2}, \ldots, a_{jn}$ so this product looks (something) like this:

$$E_{ij}A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array}.$$

The ij entry in this matrix is a_{jj}. Therefore, if $X_{ij}A = AX_{ij}$, then the ij entry in these matrices must be equal, that is

$$a_{ii} = a_{jj}.$$

This shows that for any i and j, we have $a_{ii} = a_{jj}$. That is, all diagonal entries are the same.

Note also that for $k < i$ or $k > i$, the entires a_{jk} in the second product $E_{ij}A$ must all be zero. This means that all non-diagonal entries in A are zero and A is a diagonal matrix.

We have shown $A = aI$ for some scalar a.

There is a bit of ambiguity in this problem concerning the entries in the matrices under consideration. If one takes $M_n = M_n(\mathbb{R})$ to be the $n \times n$ matrices with real entries, then we have shown

$$Z(M_n) \subset \{aI : a \in \mathbb{R}\}.$$

Another alternative would be to let R be any ring and consider $M_n = M_n(R)$. Then, we have shown

$$Z(M_n) \subset \{aI : a \in R\}.$$

I believe the general result in this case is, as Benjamin Ventimiglia points out,

$$Z(M_n) = \{aI : a \in Z(R)\}.$$

I should also like to record and credit Benjamin with the following nice phrasing:

> The matrix AE_{ij} will have the i-th column of A in the j-th column (and zeros elsewhere).
>
> The matrix E_{ij} will have the j-th row of A in the i-th row (and zeros elsewhere).