Covid 0.2
Continuity

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The objective of this document is to prompt the creation of a solid, beautiful, and elegant proof of the fact that the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is **continuous**.

1 What Continuity Means

Continuity of a function in this context means **continuity at every point**, and **continuity at a point** $x_0 \in \mathbb{R}$ means

Given $\epsilon > 0$, there is some $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \epsilon. \tag{1}$$

That's it. If you don't understand this, then there is no point in trying to give a proof. If you do understand it, then you can prove $f(x) = x^2$ is continuous (with some thought and work).

Exercise 1 *Draw the graph of $f(x) = x^2$ as a function defined on all of $\mathbb{R}$. Actually, draw the graph* **three different times***. Take the points $x_0 = 0$, $x_0 = 1$, and $x_0 = 2$. In each case, indicate x_0 in the appropriate place and indicate the interval $f(x_0) - \epsilon < v < f(x_0) + \epsilon$ in the appropriate place. What does this tell you about finding $\delta > 0$?*

2 How We Start

In some sense, to start a proof, you need to start with a positive number ϵ. In practice, you may actually write your proof with some other kind of preliminary statement as you can see that I did in my “Lecture Notes on Continuity” where I give some proofs that $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous. You don’t need to try to read those proofs carefully in order to make your own proof. In fact, some of those proofs are intentionally incorrect. (!) But you can take a look to see that the first thing I wrote did not involve $\epsilon > 0$, but you can be sure that when I started thinking about how to prove the assertion, the first thing I thought about was $\epsilon > 0$.

And that’s one of the first things you should think about too. You’ve got some number $\epsilon > 0$. More comprehensively, you have a point $x_0 \in \mathbb{R}$ and you have $\epsilon > 0$. Now, my question for you is: What are you going to think about next? What do you **need** to think about next?