Math 4317, Exam 1

1. (Assignment 1A Problem 9) Let E be any subset of the real numbers $\mathbb{R}$ and assume $u : E \to \mathbb{R}$ satisfies

$$u(x) \leq u(y) \text{ for all } x, y \in E \text{ with } x < y.$$

In this case, we say u is **monotone non-decreasing**.

(a) Let $x_0 \in \mathbb{R}$ be fixed. Show the set

$$V = \{u(x) : x \in E \text{ and } x > x_0\}$$

is **bounded below** but not necessarily **bounded above**. Note: To show V is **bounded below** you need to show there is a real number ℓ such that $\ell \leq v$ for every $v \in V$. To show V is *not necessarily* bounded above means to give an explicit example where V is not bounded above, i.e., there is no real number U such that $v \leq U$ for every $v \in V$. The number ℓ is called a **lower bound**. The number U, were such a number to exist, is called an **upper bound**.

(b) The completeness of the real numbers implies that a *nonempty* set of real numbers which is bounded below has a **greatest lower bound**, that is, a real number ℓ_0 which is a lower bound such that $\ell_0 \geq \ell$ for every lower bound ℓ. Does the set V from the previous part of this problem necessarily have a greatest lower bound? Note: If your answer is "yes," then you should prove it. If your answer is "no," then you should give an example, i.e., counterexample.

(c) If the set V from the first part of this problem has a greatest lower bound, show the set of lower bounds for V,

$$A = \{\ell : \ell \leq v \text{ for all } v \in V\},$$

is bounded above.

(d) If the set V from the first part of this problem has a greatest lower bound ℓ_0, show the least upper bound U_0 of the set A from the previous part satisfies $U_0 \leq \ell_0$.

2. (intervals; Assignment 1A Problem 10) A set $I \subset \mathbb{R}$ is an **interval** if we have $x, y \in I$ with $x < y$, then we must have

$$[x, y] = \{\xi \in \mathbb{R} : x \leq \xi \leq y\} \subset I.$$

Show that every interval has exactly one of the following ten forms:

$$\begin{aligned} &\phi \\ (-\infty, \infty) &= \mathbb{R} \\ (-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\ (-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\ (a, \infty) &= \{x \in \mathbb{R} : x > a\} \\ [a, \infty) &= \{x \in \mathbb{R} : x \geq a\} \\ (a, b) &= \{x \in \mathbb{R} : a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \end{aligned}$$

Hint: Either an interval is bounded below—or it is not. They key is to find the numbers a and/or b.

For the next two problems below, assume $u : I \to \mathbb{R}$ is a monotone non-decreasing function defined on an interval I. You may use the following fact and definition: If the least upper bound U_0 of $u((-\infty, x_0))$ and the greatest lower bound ℓ_0 of $u((x_0, \infty))$ both exist, then

$$U_0 \leq u(x_0) \leq \ell_0. \tag{1}$$

Defintion If $x_0 \in I$ and *at least one* of the inequalities in (1) is strict, we say x_0 is a **point of discontinuity** of the monotone non-decreasing function u. Note: This definition does not require that both numbers U_0 and ℓ_0 exist.

3. (Assignment 1A Problem 12) Assume $u : I \to \mathbb{R}$ is a monotone non-decreasing function defined on an interval I.
   (a) If $x_0 \in I$, when is it possible that neither the least upper bound U_0 of $u((-\infty, x_0))$ nor the greatest lower bound ℓ_0 of $u((x_0, \infty))$ exist?
   (b) If $x_0 \in I$ is a point of discontinuity of u, what are the possible relations between U_0, ℓ_0, and $u(x_0)$?

4. (Assignment 1A Problem 13) Assume $u : I \to \mathbb{R}$ is a monotone non-decreasing function defined on an interval I. Show the set of discontinuities of u is (at most) countable.

5. (Assignment 1B Problem 4) A function $\phi : G_1 \to G_2$ from one group G_1 to another G_2 is a **homomorphism** if $\phi(ab) = \phi(a)\phi(b)$ for every $a, b \in G_1$. A bijective homomorphism is called a group **isomorphism**, and two groups with a group isomorphism between them are said to be **isomorphic groups**.

   (a) Show that the **kernel**, $\ker(\phi) = \{a \in G_1 : \phi(a) = e\} = \phi^{-1}(e)$ where e is the identity element in G_2, of a homomorphism and the **image**, $\mathrm{im}(\phi) = \{\phi(a) : a \in G_1\} = \phi(G_1)$, of a homomorphism are subgroups of the groups G_1 and G_2 respectively.

   (b) If H is a subgroup of a group G, one can consider the **left cosets** of H given by

   $$aH = \{ah : h \in H\} \subset G$$

   and the **right cosets** $Ha = \{ha : h \in H\} \subset G$. A subgroup H is called **normal** if $aH = Ha$ for every $a \in G$. If H is a normal subgroup of G, then show the set of all (left) cosets $G/H = \{aH : a \in G\}$ with operation $(aH)(bH) = (ab)H$ is a group. This group G/H is called the **quotient group** of G by (the normal subgroup) H.

   (c) Show the kernel of a homomorphism is always a normal subgroup.

   (d) If $\phi : G_1 \to G_2$ is a homomorphism, then show $\mathrm{im}(\phi)$ and $G_1/\ker(\phi)$ are isomorphic groups. This is called the **first homomorphism theorem**.