Solution of Problem 9 from §1.2 of *An Introduction to Analysis* by Robert Gunning

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1 Introduction

We are asked to prove that a finite integral domain is a field.

We know that an integral domain is a ring for which $a \cdot b = 0$ implies either $a = 0$ or $b = 0$. Then, we know that an integral domain is a ring and we need its nonzero elements to form a group under multiplication for it to be a field. Notice that, since an integral domain is a ring, it has the product operation that is associative and commutative and has identity element 1. Then, the only thing we need to show is that the elements of an integral domain have multiplicative inverses. If we can show this, since associativity and identity element properties for product operation are satisfied by integral domain being a ring, we can also show that integral domain satisfies the inverse element property and therefore its nonzero elements form a group under multiplication.

2 Solution

Let I be a finite integral domain. We know that $0, 1 \in I$ since I is a ring. Precisely, we can write I such that $I = \{0, 1, x_1, x_2, ..., x_n\}$.

Claim: Consider any $x_i \neq 0$. Then, the set $\overline{I} = \{x_i, x_i x_1, x_i x_2, ..., x_i x_n\}$ is the nonzero elements of I.
Proof. Note that $\overline{I}$ has $n + 1$ elements. We have to show that all of its elements belong to I, nonzero and distinct from each other. Let $x_i, x_j \in I$ and different than

zero. Since I is closed under multiplication, any product of the form $x_i x_j$ must satisfy $x_i x_j = x_k$ for some $x_k \in I$. This implies that $\overline{I} \subseteq I$. Also note that the elements $1, x_1, x_2, ..., x_n$ are all different than zero and since x_i is also nonzero, the set $\overline{I} = \{x_i, x_i x_1, ..., x_i x_n\}$ has all nonzero elements. That is because, we know that the product of two nonzero elements is also nonzero for an integral domain since we have $a \cdot b = 0 \longrightarrow a = 0$ or $b = 0$. The contrapositive of this statement is ($a \neq 0$ and $b \neq 0$) $\longrightarrow a \cdot b \neq 0$. Finally, note that all elements of $\overline{I}$ are distinct. We can show this by the way of contradiction. If $x_i x_j = x_i x_k$ for some $x_j \neq x_k$, we have $x_i(x_j - x_k) = 0$. Because of the exactly the same argument about the property of integral domains, we have $x_i(x_j - x_k) = 0 \longrightarrow x_i = 0$ or $(x_j - x_k) = 0$. Since $x_i \neq 0$, we have $x_j - x_k = 0 \longrightarrow x_j = x_k$ which is a contradiction. Therefore, $x_i x_j \neq x_i x_k$, which means that the elements of $\overline{I}$ are distinct.

Based on this claim, we should have the element $1 \in \overline{I}$, which means that $x_i x_j = 1$ for some $x_j \in \overline{I}$. Note that this implies that $x_i x_j = x_j x_i = 1$, meaning that $x_i^{-1} = x_j$. Since x_i is arbitrarily chosen, this can be extended to any nonzero element of I. Therefore, every nonzero element of I has a multiplicative inverse and we can conclude that nonzero elements of I form a group under multiplication. We also know that I is ring, then we can conclude that it is a field as well.