Solution of Problem 1 from §1.1 of *An Introduction to Analysis* by Robert Gunning

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1 Introduction

We are asked to prove various set inclusions given in (1.4) and (1.8) in the book. These are as follows: If A, B, and C are sets, then

$$\begin{aligned} A\backslash(B \cup C) &= (A\backslash B) \cap (A\backslash C) \\ A\backslash(B \cap C) &= (A\backslash B) \cup (A\backslash C). \end{aligned} \tag{1.4}$$

If $f : A \to B$ and $X_1, X_2 \subset A$ while $Y_1, Y_2 \subset B$, then

$$\begin{aligned} &f(X_1 \cup X_2) = f(X_1) \cup f(X_2), \\ &f(X_1 \cap X_2) = f(X_1) \cap f(X_2) \text{ but it may be a proper inclusion,} \\ &f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2), \\ &f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2). \end{aligned} \tag{1.8}$$

2 Solution

A usual way to show two sets, A and B, are equal is to show one is a subset of another, $A \subset B$, and the other is a subset of the one, $B \subset A$. A usual way to show inclusion, $A \subset B$, is to take an arbitrary element $x \in A$ and then show $x \in B$.

(1.4a)

Proof of

$$A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C). \tag{1.4a}$$

$$\begin{aligned} x \in A\backslash(B \cup C) &\Longrightarrow x \in A \; x \notin B \cup C \\ &\Longrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\Longrightarrow x \in A\backslash B \text{ and } x \in A\backslash C \\ &\Longrightarrow x \in (A\backslash B) \cap (A\backslash C). \end{aligned}$$

This shows $A\backslash(B \cup C) \subset (A\backslash B) \cap (A\backslash C)$.

$$\begin{aligned} x \in (A\backslash B) \cap (A\backslash C) &\Longrightarrow x \in A\backslash B \text{ and } x \in A\backslash C \\ &\Longrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\Longrightarrow x \in A \;\text{ and } x \notin B \cup C \\ &\Longrightarrow x \in A\backslash(B \cup C). \end{aligned}$$

Thos shows the reverse inclusion $(A\backslash B) \cap (A\backslash C) \subset A\backslash(B \cup C)$ and completes the proof.

The two chains of implication in the proof above may be expressed in a more compact form using double implication:

$$\begin{aligned} x \in A\backslash(B \cup C) &\Longleftrightarrow x \in A \; x \notin B \cup C \\ &\Longleftrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\Longleftrightarrow x \in A\backslash B \text{ and } x \in A\backslash C \\ &\Longleftrightarrow x \in (A\backslash B) \cap (A\backslash C). \end{aligned}$$

This form, while correct and more compact, is generally less straightforward to read and present.

(1.4b)

Proof of

$$A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C) \tag{1.4b}$$

$$\begin{aligned} x \in A\backslash(B \cap C) &\Longrightarrow x \in A \text{ and } x \notin B \cap C \\ &\Longrightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Longrightarrow x \in A\backslash B \text{ or } x \in A\backslash C \\ &\Longrightarrow x \in (A\backslash B) \cup (A\backslash C). \end{aligned}$$

Again, each implication is reversible:

$$\begin{aligned} x \in (A\backslash B) \cup (A\backslash C) &\Longrightarrow x \in A\backslash B \text{ or } x \in A\backslash C \\ &\Longrightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Longrightarrow x \in A \text{ and } x \notin B \cap C \\ &\Longrightarrow x \in A\backslash(B \cap C). \qquad \Box \end{aligned}$$

The symbol "□" is often used at the end of a proof to indicate the end of the proof. It can be read "This completes the proof," or "This is the end of the proof." Notice the set inclusions demonstrated by these chains of implication are not stated explicitly in the proof. In this case, I decided those implications could be considered "evident to the reader," and this is okay, but if someone questions me about my proof, I will need to (and I hope I can!) remember what I had in mind.

(1.8a)

Proof of

$$f(X_1 \cup X_2) = f(X_1) \cup f(X_2). \tag{1.8a}$$

If $y \in f(X_1 \cup X_2)$, then there is some $x \in X_1 \cup X_x$ such that $y = f(x)$. If $x \in X_1$, then $y = f(x) \in f(X_1)$. If $x \in X_2$, then $y = f(x) \in f(X_2)$. Therefore, $y \in f(X_1) \cup f(X_2)$. Thus, we have shown $f(X_1 \cup X_2) \subset f(X_1) \cup f(X_2)$.

To show the reverse inclusion, let $y \in f(X_1) \cup f(X_2)$. If $y \in f(X_1)$, then there is some $x \in X_1$ such that $y = f(x)$. In particular, $y = f(x) \in f(X_1 \cup X_2)$. Similarly, if $y \in f(X_2)$, then there is some $x \in X_2$ such that $y = f(x)$. Therefore, $y = f(x) \in f(X_1 \cup X_2)$. This shows

$$f(X_1) \cup f(X_2) \subset f(X_1 \cup X_2)$$

and establishes (1.8a). □

(1.8b)

Proof of

$$f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2) \text{ but it may be a proper inclusion.} \tag{1.8b}$$

If $y \in f(X_1 \cap X_2)$, then there is some $x \in X_1 \cap X_x$ such that $y = f(x)$. Since[1] $x \in X_1$, we know $y = f(x) \in f(X_1)$. Similarly, since $x \in X_2$, we also know $y = f(x) \in f(X_2)$. This means $y = f(x) \in f(X_1) \cap f(X_2)$. We have shown the inclusion $f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2)$. □

To show that the reverse inclusion need not hold let $A = \{a, b\}$ and $B = \{0\}$ with $f : \{a, b\} \to \{0\}$ by $f(a) = f(b) = 0$. Taking $X_1 = \{a\}$ and $X_2 = \{b\}$, we find

$$f(X_1) = \{0\} = f(X_2) \qquad \text{so} \qquad f(X_1) \cap f(X_2) = \{0\}.$$

On the other hand, $X_1 \cap X_2 = \phi$, so while

$$f(X_1 \cap X_2) = \phi \subset \{0\} = f(X_1) \cap f(X_2),$$

we do not have equality, so (in this example)

$$f(X_1 \cap X_2) \subsetneqq f(X_1) \cap f(X_2).$$

Some mathematicians also end examples and counterexamples with the □ symbol to indicate the end. (I, and many others, reserve □ for the end of proofs.)

(1.8c)

Proof of

$$f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2). \tag{1.8c}$$

If $x \in f^{-1}(Y_1 \cup Y_2)$, then $f(x) \in Y_1$ or $f(x) \in Y_2$. In the former case, $x \in f^{-1}(Y_1)$, and in the latter case $x \in f^{-1}(Y_2)$. This means $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$ and

$$f^{-1}(Y_1 \cup Y_2) \subset f^{-1}(Y_1) \cup f^{-1}(Y_2).$$

[1]This sentence and the next one are perhaps more detail than one needs in this proof, but I'm just including "every" detail. Again, if one is asked about the details, it's nice to be able to supply them. As a consequence, sometimes it's nice to include them. You are encouraged to read the proof with these two sentences omitted, and see which way you think is easier to read. Sometimes "less is more," more or less.

To show the reverse inclusion, assume $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$. If $x \in f^{-1}(Y_1)$, then $f(x) \in Y_1 \subset Y_1 \cup Y_2$. Therefore,

$$x \in f^{-1}(Y_1 \cup Y_2).$$

Similarly, if $x \in f^{-1}(Y_2)$, then $f(x) \in Y_2 \subset Y_1 \cup Y_2$, and again we have

$$x \in f^{-1}(Y_1 \cup Y_2).$$

We have shown

$$f^{-1}(Y_1) \cup f^{-1}(Y_2) \subset f^{-1}(Y_1 \cup Y_2). \qquad \square$$

(1.8d)

Proof of

$$f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2). \tag{1.8d}$$

$$\begin{aligned} x \in f^{-1}(Y_1 \cap Y_2) &\iff f(x) \in Y_1 \cap Y_2 \\ &\iff x \in f^{-1}(Y_1) \text{ and } x \in f^{-1}(Y_2) \\ &\iff x \in f^{-1}(Y_1) \cap f^{-1}(Y_2). \qquad \square \end{aligned}$$