Assignment 1B Problem 13
Undergraduate Analysis, Spring 2020

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I composed this solution after a correct and insightful solution was presented in class by Eliot Xing during the Spring 2020 semester. After seeing Eliot's solution, I felt that the way I was thinking about the problem was significantly different enough to write down my approach and make it available.

Here we are given a non-decreasing function $u : I \to \mathbb{R}$ defined on an interval I. The functions $u_-(x) = \sup u((-\infty, x))$ and $u_+(x) = \inf((x, \infty))$ have been previously defined for points $x \in I$ for which there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < x < b$ and $(a, b) \subset I$. From these the non-negative jump function

$$S(x) = u_+(x) - u_-(x) \tag{1}$$

has been defined. This function has also been extended to all of I by noting that if $x \in I$ is a point at which S is not defined then there are exactly three possibilities:

1. $x = \min I$ and there is some $b \in \mathbb{R}$ such that $x < b$ and $[x, b) \subset I$. In this case, we can set $S(x) = u_+(x) - u(x)$.

2. $x = \max I$ and there is some $a \in \mathbb{R}$ such that $a < x$ and $(a, x] \subset I$. In this case, we can set $S(x) = u(x) - u_-(x)$.

3. $I = \{x\} = [x, x]$ is a singleton. In this case, we can set $S(x) = 0$, but this is obviously not a very interesting case.

Let me mention (for future reference) that a somewhat more general approach may be taken in which $u_-(x)$ is defined by $u_-(x) = u(x)$ in the first case above, $u_+(x) = u(x)$ in the second case, and $u_-(x) = u_+(x) = u(x)$ in the last (uninteresting) case. Then, formula (1) may be used in all situations.

With these notions in place, we are asked to show the following:

(a) If $S(x) = 0$, then for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$\left.\begin{array}{c}|\xi - x| < \delta \\ \xi \in I\end{array}\right\} \qquad \Longrightarrow \qquad |u(\xi) - u(x)| < \epsilon. \tag{2}$$

(b) Given $x \in I$ such that for any $\epsilon > 0$ there is some δ for which (2) holds, then $S(x) = 0$.

Proof of (a):

I'm going to change the notation a little bit and call the point in question x_0 instead of x. This will free up the symbol x and we can avoid using ξ. Accordingly, we begin with the hypothesis

$$S(x_0) = 0. \tag{3}$$

Also, let us assume (BWOC) the assertion associated with (2) does not hold for $x = x_0$. For convenience, clarity, and future reference, let's restate that condition explicitly:

$$\begin{array}{c}\text{For any } \epsilon > 0, \text{ there is some } \delta > 0 \text{ such that} \\[1em] \left.\begin{array}{c}|x - x_0| < \delta \\ x \in I\end{array}\right\} \qquad \Longrightarrow \qquad |u(x) - u(x_0)| < \epsilon.\end{array} \tag{4}$$

If (4) does not hold, there is a fixed $\epsilon = \epsilon_0 > 0$ and there exist points $x_j \in I$ for $j = 1, 2, 3, \ldots$ with

$$|x_j - x_0| < \frac{1}{j} \qquad \text{and} \qquad |u(x_j) - u(x_0)| \geq \epsilon_0 > 0. \tag{5}$$

Clearly, $x_j \neq x_0$, so one of the sets

$$\Gamma_- = \{j \in \mathbb{N} : x_j < x_0\} \qquad \text{or} \qquad \Gamma_+ = \{j \in \mathbb{N} : x_j > x_0\}$$

has cardinality $\aleph_0$, that is, one of these sets of indices is infinite and *contains arbitrarily large natural numbers.*

Note that if $j \in \Gamma_-$, then $x_j < x_0$ and $u(x_j) < u(x_0)$. It follows then that

$$|u(x_j) - u(x_0)| = u(x_0) - u(x_j) \geq \epsilon_0 \qquad \text{or} \qquad u(x_j) \leq u(x_0) - \epsilon_0. \tag{6}$$

Similarly, if $j \in \Gamma_+$, then $x_0 < x_j$ and $u(x_0) < u(x_j)$, so

$$u(x_j) \geq u(x_0) + \epsilon_0.$$

If there are infinitely many indices in Γ_-, then for any $x < x_0$, there is some $x_j < x_0$ with $1/j < x_0 - x$ so that $x < x_j < x_0$. Thus, in view of (6)

$$u(x) \leq u(x_j) \leq u(x_0) - \epsilon_0.$$

It follows from the definition of the supremum in $u_-(x_0)$ that $u_-(x_0) \leq u(x_0) - \epsilon_0$. Since we have shown previously that $u(x_0) \leq u_+(x_0)$, this means

$$S(x_0) = u_+(x_0) - u_-(x_0) \geq u(x_0) - [u(x_0) - \epsilon_0] = \epsilon_0 > 0.$$

This contradicts the hypothesis $S(x_0) = 0$.

Similarly, if $\#\Gamma_+ = \aleph_0$, then for any $u_+(x_0) \geq u(x_0) + \epsilon_0$ and $S(x_0) > \epsilon_0 > 0$ contradicting (3).

These contradictions establish the that (4) holds if $S(x_0) = 0$. $\square$

Proof of (b):

Conversely, let us begin with the hypothesis (4) but assume (BWOC) that

$$S(x_0) = u_+(x_0) - u_-(x_0) > 0. \tag{7}$$

Then, since we have shown previously that

$$u_-(x_0) \leq u(x_0) \leq u_+(x_0),$$

we must have

$$u_-(x_0) < u(x_0) \qquad \text{or} \qquad u(x_0) < u_+(x_0).$$

Consider the case $u_-(x_0) < u(x_0)$. Then

$$\epsilon_0 = u(x_0) - u_-(x_0) > 0.$$

In particular, this means

$$I_- = \{x \in I : x < x_0\} = (-\infty, x_0) \cap I \neq \phi$$

and for every $x \in I$ with $x < x_0$ we have

$$u(x) \le u_-(x_0) = u(x_0) - [u(x_0) - u_-(x_0)] = u(x_0) - \epsilon_0.$$

It follows that for any $\delta > 0$, there is some $x \in I_-$ with $|x - x_0| = x_0 - x < \delta$ and

$$|u(x) - u(x_0)| = u(x_0) - u(x) \ge \epsilon_0.$$

This contradicts (4).

Similarly, if $u(x_0) < u_+(x_0)$, then we may set $\epsilon_0 = u_-(x_0) - u(x_0) > 0$, and for any $\delta > 0$, there is some $x > x_0$ with $x \in I$ and $|x - x_0| = x - x_0 < \delta$ such that

$$|u(x) - u(x_0)| = u(x) - u(x_0) \ge \epsilon_0.$$

Again, we have contradicted our hypothesis (4), and these contradictions show that $S(x_0) = 0$ according to the assertion of (b). $\square$