Assignment 2: The 1-D Heat Equation
Due Tuesday September 19, 2023

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Problem 1 *(Haberman 1.4.4) Assume heat conduction is modeled in a thin metal rod by*

$$u_t = (ku_x)_x \qquad \text{on} \qquad (0, \ell) \times (0, \infty)$$

where $k = k(x)$ depends on position. If both ends of the rod are modeled as insulated, show the total heat energy in the rod must be constant (as a function of time).

Problem 2 *(Haberman 1.4.5) Assume heat conduction is modeled in a thin metal rod by*

$$u_t = ku_{xx} \qquad \text{on} \qquad (0, \ell) \times (0, \infty)$$

where the diffusivity k is a constant. Give an expression for the temperature $U(\ell)$ of an equilibrium solution $U = U(x)$ with $U(0) = T_1$ and $U_x(0) = r_1$.

Problem 3 *(Haberman 1.4.6) If heat conduction in a thin metal rod is modeled by the forced 1-D heat equation with nonzero constant source term Q, and both ends are modeled as insulated, prove there can be no equilibrium solution*

$$U(x) = \lim_{t \nearrow \infty} u(x, t).$$

Problem 4 *(Haberman 1.4.6) Under the assumption(s) of the previous problem calculate the total thermal energy*

$$\int_0^\ell c\rho u \, dx$$

as a function of time. Hint: Your answer should depend on the initial temperature $u(x, 0)$.

Problem 5 *Use Fourier's law to determine an appropriate boundary condition on an n-dimensional region R corresponding to heat conduction in R with* **insulated boundary**.

Problem 6 *Interpret the integral*

$$\int_{\partial R} \vec{\phi} \cdot \vec{n}$$

in one space dimension.

Problem 7 *Find a solution $u = u(x,t)$ of the 1-D diffusion PDE*

$$u_t = u_{xx} \tag{1}$$

having the form

$$u(x,t) = f(t) \sin x$$

for some real valued function $f = f(t)$. Assuming equation (1) is derived to model a heat conduction problem along a spatial interval $0 \leq x \leq \ell$ with specific heat capacity $c = 1$ and conductivity $K = 1$, determine the rate and direction of heat flow at each boundary point $x = 0$ and $x = \ell$.

Problem 8 *Solve the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell,\ t > 0 \\ u(0,t) = 0 = u(\ell,t), & t > 0 \\ u(x,0) = \sin(\pi x/\ell), & 0 \leq x \leq \ell. \end{cases}$$

Hint: See Problem 7 above.

Problem 9 *Take $\ell = 1$ and use numerical software to plot your solution to Problem 8 as the graph of a function of two variables (in three dimensions).*

Problem 10 *(Haberman 1.4.9) Integrate the diffusion equation*

$$c\rho u_t = K u_{xx}$$

to obtain the integral conservation law

$$\frac{d}{dt} \int_0^\ell c\rho u \, dx = K[u_x(\ell,t) - u_x(0,t)].$$

Assume c, ρ, and K are constant and explain clearly the regularity assumptions you need for u to satisfy.