Assignment 5: Laplace's Equation
Due Tuesday October 24, 2023

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Problem 1 (uniqueness for solutions of the heat equation) Let u and v be solutions of the inital/boundary value problem

$$\begin{cases} u_t = \Delta u + f, & (\mathbf{x}, t) \in U \times (0, \infty) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in U \\ u(\mathbf{x}, t) = u_0(\mathbf{x}), & x \in \partial U, t > 0 \end{cases}$$

on the open, bounded, connected spatial domain $U \subset \mathbb{R}^n$ with smooth boundary ∂U. Complete the following to show $u \equiv v$.

(a) Consider the difference $w = u - v$. Find an initial/boundary value problem satisfied for w.

(b) Consider the square

$$A(t) = \int_U w^2$$

of the L^2 norm of w, and show $A'(t) \leq 0$. Hint: Differentiate under the integral sign, use the equation, and apply the divergence theorem. Hint hint: Show

$$\operatorname{div}(wDw) = |Dw|^2 + w\Delta w.$$

(c) Conclude $w \equiv 0$. Hint: The IVP $A' = 0$, $A(0) = 0$ has a unique solution.

Problem 2 (1-D heat equation, infinite propogation speed) Consider the initial/boundary value problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in (0,3) \times (0,\infty) \\ u(x,0) = \chi_{[1,2]}(x), & x \in (0,3) \\ u(0,t) = 0 = u(3,t), & t > 0 \end{cases}$$

(a) Solve the problem using the method of separation of variables and Fourier expansion.

(b) Show $u(x,t) > 0$ for $0 < x < 3$ and $t > 0$.

(c) Plot the solution. Technically, this means plot a Fourier approximation of the solution using an approximation involving some reasonably large number of terms in the Fourier series.

(d) Why does the assertion of part **(b)** illustrate infinite speed propagtion?

Problem 3 (Laplace's equation) Find all separated variables solutions of the boundary value problem

$$\begin{cases} \Delta u = 0, & (x,y) \in (0,L) \times (0,M) \\ u(x,0) = 0 = u(x,M), & x \in (0,L) \end{cases}$$

where $L, M > 0$.

Problem 4 (Laplace's equation) Choose specific positive values for L and M in the last problem.

(a) Plot (the graph of) one of your separated variables solutions obtained in Problem 3.

(b) Give the full boundary value problem for the solution plotted in part **(a)**.

(c) Plot (the graph of) a superposition of two of your separated variables solutions obtained in Problem 3.

(d) Give the full boundary value problem for the solution plotted in part **(c)**.

Problem 5 (Homogeneous boundary conditions on a rectangle) Solve the boundary value problem for Laplace's equation

$$\begin{cases} \Delta u = 0, & (x,y) \in (0,2) \times (0,\pi) \\ u(x,0) = 0 = u(x,\pi), & x \in (0,2) \\ u(0,y) = 0, & y \in (0,\pi) \\ u(2,y) = \sin y, & y \in (0,\pi). \end{cases}$$

Problem 6 (Uniqueness of solutions for Laplace's equation and Poisson's equation) Let u and v be solutions of the boundary value problem

$$\begin{cases} \Delta u = f, & \mathbf{x} \in U \\ u(\mathbf{x}) = u_0, & \mathbf{x} \in \partial U \end{cases}$$

where U is an open bounded connected domain in $\mathbb{R}^n$ with smooth boundary. Complete the following to show $u \equiv v$.

(a) Let $w = u - v$ and find a boundary value problem for w.

(b) Show the L^2 norm of w is zero. Hint: Divergence theorem. Hint: See Problem 1 above.

Problem 7 (Laplace's equation on a rectangle) Solve the boundary value problem

$$\begin{cases} \Delta u = 0, & (x,y) \in (0,2) \times (0,\pi) \\ u(x,0) = \sin(k\pi x/2), & x \in (0,2) \\ u(x,\pi) = \sin(\ell\pi x/2), & x \in (0,2) \\ u(0,y) = \sin(my), & y \in (0,\pi) \\ u(2,y) = \sin(ny), & y \in (0,\pi) \end{cases}$$

where k, ℓ, m and n are positive integers. Hint: Solve four problems like the one in Problem 5 and add the four solutions you get.

Problem 8 (series) Find a series solution of the boundary value problem

$$\begin{cases} \Delta u = 0, & (x,y) \in (0,2) \times (0,2\pi) \\ u(x,0) = 0 = u(x,2\pi), & x \in (0,2) \\ u(0,y) = 0, & y \in (0,2\pi) \\ u(2,y) = \pi - |y - \pi|, & y \in (0,2\pi). \end{cases}$$

Problem 9 (maximum value)

(a) Plot your solution from Problem 8.

(b) Based on your plot (try to) identify the maximum value taken by the solution u.

(c) Find the maximum value of the solution plotted in Problem 4 part **(a)**.

Problem 10 (weak maximum principle) Complete the following to prove a solution of Laplace's equation cannot have an interior maximum with value greater than the boundary maximum.

(a) Assume $u \in C^2(U) \cap C^0(\overline{U})$ solves Laplace's equation with

$$m = \max_{\mathbf{x} \in \partial U} u(\mathbf{x}) < M = \max_{\mathbf{x} \in \overline{U}} u(\mathbf{x}) = u(\mathbf{p}).$$

Show the function $\phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi(\mathbf{x}) = M - \epsilon |\mathbf{x} - \mathbf{p}|^2$$

satisfies

$$\phi(\mathbf{x}) > u(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial U$$

if $\epsilon > 0$ is small enough.

(b) Calculate $\Delta \phi$.

(c) Show that for some $\delta \geq 0$ the function $\psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(\mathbf{x}) = \phi(\mathbf{x}) + \delta$$

satisfies

$$\psi(\mathbf{x}) \geq u(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \overline{U}.$$

and $\psi(\mathbf{q}) = u(\mathbf{q})$ for some $\mathbf{q} \in U$.

(d) Show $\Delta u(\mathbf{q}) \leq \Delta \psi(\mathbf{q})$. (What does this tell you?)