Assignment 6: Weak C^1 Solutions
Due Tuesday November 7, 2023

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I would like to discuss certain **weak C^1 solutions** of the 1-D heat equation. In order to motivate this discussion, I'm going to start with weak C^1 solutions of a certain ordinary differential equation (ODE). Consider the second order ordinary differential equation

$$\frac{d^2y}{dx^2} = y'' = f(x). \tag{1}$$

If the function f is continuous, then this ODE is quite easy to solve in general. Specifically, let us consider (1) for $a \le x \le b$ and assume $f \in C^0[a, b]$. Then

$$y(x) = c_1 x + c_2 + \int_a^x \left(\int_a^\xi f(t)\, dt \right) d\xi \tag{2}$$

satisfies $y \in C^2[a, b]$ for all constants c_1 and c_2, and

$$y'' = f, \tag{3}$$

That is, y solves the ODE (1). The first five problems of this assignment concern just what we have written above about the simple ODE (3) and its solution (2).

Problem 1 (the meaning of $C^1[a, b]$) Let $f \in C^0[a, b]$, which you should recall just means f is **continuous** on the closed interval

$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}.$$

and consider the function $y : [a, b] \to \mathbb{R}$ given by the formula (2).

(a) Various definitions of continuity are possible. Here is one: For any $x_0 \in [a, b]$ and any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|f(x) - f(x_0)| < \epsilon \qquad \text{whenever } x \in [a, b] \text{ and } |x - x_0| < \delta.$$

Reexpress this condition in terms of limits.

(b) Recall that the definition

$$y'(x_0) = \lim_{h \to 0} \frac{y(x_0 + h) - y(x_0)}{h} \tag{4}$$

of the derivative $y'(x_0)$ generally requires the value $y(x)$ of the function y to be defined for x in some open interval

$$(x_0 - \delta, x_0 + \delta) = \{x \in \mathbb{R} : x_0 - \delta < x < x_0 + \delta\}$$

where δ is some positive number. Explain why this is the case.

(c) What goes wrong if you try to compute $y'(a)$ using the formula (2)?

(d) As a consequence of your consideration of parts **(a)**, **(b)**, and **(c)** above, you should appreciate the assertion that it is easier to define the set of functions $C^1(a, b)$ than the set $C^1[a, b]$ though this problem does not arise in connection with $C^0[a, b]$. One possibility is to define $C^1[a, b]$ to be the collection of functions $u \in C^0[a, b]$ for which the restriction

$$u\Big|_{(a,b)} \in C^1(a, b)$$

and the **left and right derivatives**

$$u'(b^-) = \lim_{h \nearrow 0} \frac{u(b + h) - u(b)}{h} \qquad \text{and} \qquad u'(a^+) = \lim_{h \searrow 0} \frac{u(a + h) - u(a)}{h}$$

exist (as well-defined real numbers) and the function $v : [a, b] \to \mathbb{R}$ by

$$v(x) = \begin{cases} u'(a^+), & x = a \\ u'(x), & a < x < b \\ u'(b^-), & x = b \end{cases}$$

satisfies $v \in C^0[a, b]$. Show this definition of $u \in C^1[a, b]$ is equivalent to

$$C^1[a, b] = \left\{ u \in C^0[a, b] : \text{there exists some } w \in C^1(\mathbb{R}) \text{ with } w\Big|_{[a,b]} = u \right\}.$$

Problem 2 (classical solution) In view of Problem 1 above, given $u \in C^1[a, b]$ we denote the common value(s)

$$u'(a^+) = w'(a) \qquad \text{and} \qquad u'(b^-) = w'(b)$$

where $w \in C^1(\mathbb{R})$ is any extension of u by $u'(a)$ and $u'(b)$ respectively. With this convention, verify the following assertions stated above concerning the solution (2):

(a) $y \in C^2[a, b] = \{u \in C^1[a, b] : u' \in C^1[a, b]\}$.

(b) $y''(x) = f(x)$ for $a \le x \le b$.

Problem 3 (uniqueness of classical solutions) Show that given $f \in C^0[a, b]$, every solution $y \in C^2[a, b]$ can be written in the form (2).

Problem 4 (initial value problem) Solve the initial value problem

$$\begin{cases} y'' = f, & a \le x \le b \\ y(x_0) = y_0, & \\ y'(x_0) = v_0, & \end{cases} \tag{5}$$

where $f \in C^0[a, b]$, $x_0 \in [a, b]$ and $y_0, v_0 \in \mathbb{R}$ are given.

Problem 5 (two point boundary value problem) Solve the two point boundary value problem

$$\begin{cases} y'' = f, & a \le x \le b \\ y(a) = y_a, & \\ y(b) = y_b, & \end{cases} \tag{6}$$

where $f \in C^0[a, b]$ and $y_a, y_b \in \mathbb{R}$ are given.

The next four problems attempt to address the situation in which one wishes to consider the ODE (1)—or something like it—involving a function $f \notin C^0[a, b]$. As a primary example, consider $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \chi_{[1,2]}(x) = \begin{cases} 1, & x \in [1, 2] \\ 0, & x \notin [1, 2]. \end{cases} \tag{7}$$

For this function the formula (2) still defines a function $y \in C^1(\mathbb{R})$.

Problem 6 (example properties of a weak solution) Consider $y : [0, 3] \to \mathbb{R}$ by

$$y(x) = c_1 x + c_2 + \int_0^x \left(\int_0^\xi \chi_{[1,2]}(t)\, dt \right) d\xi. \tag{8}$$

(a) Calculate $y'(x)$ for $x \in [0, 3]$.

(b) Plot y' on the interval $0 \le x \le 3$.

(c) Show $y' \in C^0[0, 3] \backslash C^1[0, 3]$.

(d) Show $y''(1)$ and $y''(2)$ do not exist.

As a consequence of Problem 6 we see that for the particular function f given in (7) not only is it true that $y \notin C^2[0, 3]$ when y is given by (2 or 8) but it does not make sense to write the equation(s) (1 or 3). Since the ODE (1 or 3) does not make any immediate sense at least in some cases when $f \notin C^0[a, b]$, I propose to **replace the ODE itself** with a different formulation/condition. Before giving this formulation, let me mention briefly some details concerning the general class of inhomogeneities f to which the formulation applies. We consider

$$f \in L^1(a, b) \cap L^\infty(a, b). \tag{9}$$

An alternative (more general) development could be given by $f \in L^1_{loc}(a, b)$, but the collection of functions in (9) contains most of the functions we will discuss and affords some simplifications. In order to explain (roughly) the functions contained in $L^1(a, b)$ and $L^\infty(a, b)$, it is natural to introduce a better kind of integration. This is **Lebesgue integration**, and you can think of it as having many if not most of the properties you are used to with Riemann integration, but it is a kind of integration which can be applied to many functions which are not Riemann integrable. Without giving the details of the construction of Lebesgue integration, I will simply introduce a notation to let you know it is being used: Let

$$\int_{(a,b)} f \tag{10}$$

denote the Lebesgue integral of a Lebesgue integrable function $f : (a, b) \to \mathbb{R}$. If f happens to be Riemann integrable, then we can change the notation to the familiar one

$$\int_a^b f(x)\, dx. \tag{11}$$

Every Riemann integrable function f is Lebesgue integrable, and the value given in (11) in such a case will match the value given by the Lebesgue integral in (10). With this in mind,

$$L^1(a,b) = \left\{ f : \int_{(a,b)} |f| < \infty \right\}.$$

Thus, $L^1(a,b)$ is the collection of real valued Lebesgue integrable functions f on the interval (a,b) with absolute value having finite integral. This collection is often called the collection of **finite Lebesgue integrable functions**.

The collection $L^\infty(a,b)$ is slightly more complicated. Perhaps we can start with the name. These are the **essentially bounded measurable functions**. Thus, the intersection $L^1(a,b) \cap L^\infty(a,b)$ consists of the **essentially bounded finite Lebesgue integrable functions**. But what does it mean to be essentially bounded? The answer is that there is some constant $M > 0$ for which the set

$$\{x \in (a,b) : |f(x)| > M\}$$

has measure zero. Of course, I should tell you what it means for a set to have measure zero. A set $A \subset \mathbb{R}$ has **measure zero** if for every $\epsilon > 0$ there is some sequence of open intervals $I_j = (a_j, b_j)$, $j = 1, 2, 3, \ldots$ for which

$$A \subset \cup_{j=1}^\infty I_j \qquad \text{and} \qquad \sum_{j=1}^\infty (b_j - a_j) < \epsilon.$$

An example of such a function is $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

You can check that this function is not Riemann integrable. It turns out that it is Lebesgue integrable with $\int f = 0$. Thus, in $L^1(\mathbb{R})$ this function cannot be distinguished from the zero function. Its values are universally bounded by $M = 1$ and essentially bounded by $M = 0$. We now wish to make sense of and ODE like "$y'' = f$" where f is this function. Here is the definition:

Definition 1 Given $f \in L^1(a,b) \cap L^\infty(a,b)$, we say $y \in C^1[a,b]$ is a **weak** C^1 **solution** of the ODE "$y'' = f$" if

$$-\int_{(a,b)} y' \, \phi' = \int_{(a,b)} f \, \phi \qquad \text{for every } \phi \in C_c^\infty(a,b). \tag{12}$$

Notice the condition (12) makes sense for every $y \in C^1[a,b]$.

Problem 7 (weak solutions with regularity) If $y \in C^2[a,b]$ satisfies (12) and $f \in C^0[a,b]$, then show $y'' = f$. In this case, we say y is a **classical solution**.

The formula (2) can be generalized to the situation when $f \in L^1(a,b) \cap L^\infty(a,b)$ and reads

$$y(x) = c_1 x + c_2 + \int_{(a,x)} \left(\int_{(a,\xi)} f \right). \tag{13}$$

We have already seen above, even in the case when f is Riemann integrable but discontinuous, that this formula does not always produce a classical solution.

Problem 8 (example of a weak solution) Taking $f = \chi_{[1,2]}$ as in (8), show the formula (8 or 13) considered in Problem 6 yields a weak C^1 solution of the ODE "$y'' = \chi_{[1,2]}$."

Problem 9 (uniqueness of weak C^1 solutions) Let $u, y \in C^1[a,b]$ be weak C^1 solutions of the ODE "$y'' = f$" for some function $f \in L^1(a,b) \cap L^\infty(a,b)$.

(a) Show that for some constant $c_1 \in \mathbb{R}$ there holds $u'(x) = y'(x) + c_1$ for $x \in [a,b]$.

(b) What can you say about the uniqueness of weak C^1 solutions satisfying initial conditions $y(x_0) = y_0$ and $y'(x_0) = v_0$ for some $x_0 \in [a.b]$ and $y_0, v_0 \in \mathbb{R}$?

(c) What can you say about the uniqueness of weak C^1 solutions satisfying two point boundary conditions $y(a) = y_a$ and $y(b) = y_b$ for some $y_a, y_b \in \mathbb{R}$?

Hint: Part **(a)** is somewhat tricky. If you are stuck, here are some hints in increasing detail.

(i) Let $w = u - y$ and try to show w' is constant.

(ii) Show a function $\psi \in C_c^\infty(a,b)$ satisfies $\psi = \phi'$ for some $\phi \in C_c^\infty(a,b)$ if and only if $\int \psi = 0$.

(ii) Take an arbitrary $\mu \in C_c^\infty(a,b)$ and build a function $\psi \in C_c^\infty(a,b)$ having the property that $\psi = \phi'$ for some $\phi \in C_c^\infty(a,b)$.

(iii) Let $\eta \in C_c^\infty(a,b)$ be fixed with $\int \eta = 1$. Consider $\psi = \mu - c\eta$ for an appropriate constant c.

(iv) Apply the fundamental lemma of vanishing integrals using the test function μ.

Weak C^1 solutions of the 1-D heat equation

Finally, we address the main topic of this assignment:

Definition 2 (weak C^1 solution of the heat equation) Given $f \in L^1(0,\ell) \cap L^\infty(0,\ell)$, a function $u \in C^1([0,\ell] \times [0,\infty))$, is said to be a **weak C^1 solution** of the spatially forced 1-D heat equation "$u_t = u_{xx} + f$" if

$$\int_{(0,\ell)} u_t \, \phi = -\int_{(0,\ell)} u_x \, \phi' + \int_{(0,\ell)} f \, \phi \qquad \text{for every } \phi \in C_c^\infty(0,\ell).$$

Problem 10 (example weak C^1 solution of the heat equation) Consider the problems

$$\begin{cases} \text{“}u_t = u_{xx} + \chi_{[1,2]},\text{”} & \text{on } (0,3) \times (0,\infty) \\ u(0,t) = 0 = u(\ell,t), & t > 0 \\ u(x,0) = 0, & 0 < x < 3, \end{cases} \tag{14}$$

and

$$\begin{cases} v_t = v_{xx}, & \text{on } (0,3) \times (0,\infty) \\ v(0,t) = 0 = v(\ell,t), & t > 0 \\ v(x,0) = v_0, & 0 < x < 3, \end{cases} \tag{15}$$

(a) Let $y \in C^1[0,3]$ be the unique weak C^1 solution of the ODE "$y'' = \chi_{[0,2]}$" satisfying the two boundary point conditions $y(0) = 0 = y(3)$.

(b) Make an appropriate choice of the initial temperature v_0 so that the following theorem (which we will not prove) applies and you can complete part **(c)** below.

> **Theorem 1** If $v_0 \in C^1[0,3]$ with $v_0(a) = 0 = v_0(b)$, then the problem (15) has a unique solution $u \in C^1([0,3] \times [0,\infty)) \cap C^\infty((0,3) \times (0,\infty))$.

(c) For an appropriate choice of v_0 in part **(b)** above and v the unique classical solution (15) given by Theorem 1, show $u = v - y$ is a weak C^1 solution of the initial/boundary value problem (14).