Final Assignment: 1D Wave Equation
Generalized d'Alembert formulas and other topics
Due Thursday December 14, 2023

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The wave equation in one space dimension has the form

$$u_{tt} = u_{xx}.$$

As we have discussed, if this equation is considered for a function u with domain $\mathbb{R} \times [0, \infty)$ the natural Cauchy data has the form

$$\left\{ \begin{array}{ll} u(x,0) = u_0(x), & x \in \mathbb{R} \\ u_t(x,0) = v_0(x), & x \in \mathbb{R} \end{array} \right.$$

along the Cauchy manifold corresponding to time $t = 0$, that is the x-axis. The natural (simplest) regularity assumption for a solution in this formulation is $u \in C^2(\mathbb{R} \times [0, \infty))$ and under this assumption one obtains d'Alembert's solution

$$u(x,t) = \frac{1}{2}[u_0(x-t) + u_0(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} v_0(\xi)\, d\xi \tag{1}$$

which may be considered a kind of starting point for the development below. There are several topics that will be addressed, but let us think initially about replacing the real line $\mathbb{R}$ with a finite spatial interval $[a, b] \subset \mathbb{R}$. The resulting domain $[a, b] \times [0, \infty)$ is indicated in Figure 1. Note this domain has interior $U = (a, b) \times (0, \infty)$.

First note that d'Alembert's formula (1) is going to have trouble in the case of a bounded spatial domain because the arguments $x - t$ and $x + t$ may not fall within the domain of definition of the functions u_0 and v_0 providing the initial or Cauchy data as these will only naturally be assumed to be defined on the spatial interval $[a, b]$. There is still a natural domain

$$W = \left\{ (x,t) \in U : 0 < t < \frac{b-a}{2} - \left| x - \frac{a+b}{2} \right| \right\}$$

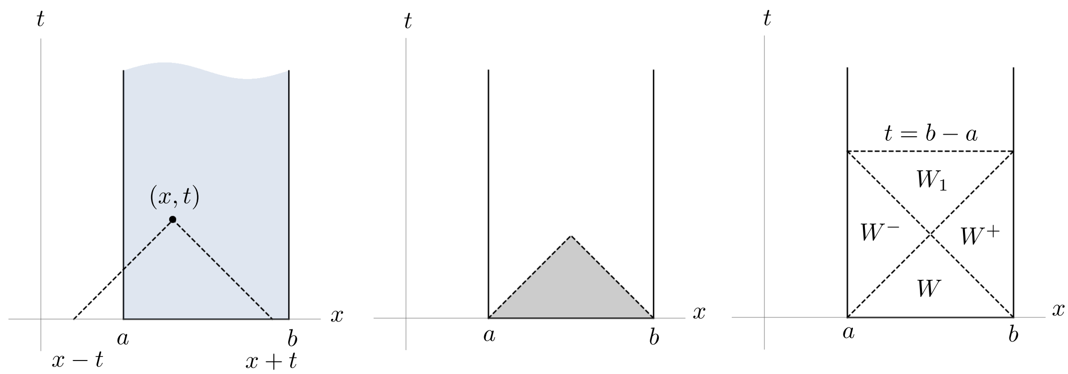

Figure 1: General 1-D wave domain on a finite spatial interval $[a, b]$. Natural subdomain for d'Alembert solution (center). Extension subdomains and natural first truncation (right).

as indicated in Figure 1 (center) where (1) is a well-defined formula and can be expected to hold. In particular, if the initial/Cauchy data satisfies

$$u_0 \in C^2(a,b) \cap C^0[a,b] \qquad \text{and} \qquad v_0 \in C^1(a,b) \cap C^0[a,b],$$

then formula (1) defines a function in $C^2(W)$ which solves $u_{tt} = u_{xx}$ classically. We have noted that the formula (1) can make sense for functions u_0 and v_0 having considerably less regularity and may still be considered to define some kind of solution of the wave equation on W. We will discuss a notion of weak solution below which can partially make sense of this kind of assertion.

For the moment we may think in terms of introducing additional **boundary conditions** at the points $x = a$ and $x = b$. The natural conditions constituting Cauchy data for the PDE on $x = a$ or $x = b$ would involve prescribing the value of the function u and the value of some **transverse derivative** of u. In this case, with vertical boundary lines $x = a$ and $x = b$, the easiest choice is to prescribe the spatial derivative u_x. Thus we arrive at an initial/boundary value problem for the wave equation having the intimidating form

$$\left\{\begin{array}{rcll} u_{tt} & = & u_{xx}, & (x,t)\in U \\ u(x,0) & = & u_0(x), & x\in(a,b) \\ u_t(x,0) & = & v_0(x), & x\in(a,b) \\ u(a,t) & = & u_0^-(t), & t>0 \\ u_x(a,t) & = & \sigma_0^-(t), & t>0 \\ u(b,t) & = & u_0^+(t), & t>0 \\ u_x(b,t) & = & \sigma_0^+(t), & t>0. \end{array}\right. \tag{2}$$

The symbol σ here is used because it gives the value of a **s**patial derivative.

My first objective will be to consider finding an extension of the d'Alembert formula to the domain

$$W^- = \{(x,t)\in U : -a+x<t<b-x\}$$

as indicated in Figure 1 (right). At each stage I will state results or questions and leave the corresponding result or question concerning the extension to

$$W^+ = \{(x,t)\in U : b-x<t<-a+x\}$$

as a problem. Ultimately, we should like to obtain an extension to the entire truncated domain $U_1 = W\cup W^-\cup W^+\cup W_1$ indicated in Figure 1 along with appropriate Cauchy data on the time truncation line $t=b-a$ of the form

$$\left\{\begin{array}{ll} u(x,b-a)=u_1(x), & x\in\mathbb{R} \\ u_t(x,b-a)=v_1(x), & x\in\mathbb{R} \end{array}\right.$$

for some appropriate Cauchy data functions u_1 and v_1. Then the entire process may be repeated to extend the solution u. We will not get to the domain

$$W_1 = \left\{(x,t)\in U : \frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|<t<b-a\right\}$$

in general, but we will consider this extension in a special case at the end. You should also understand by the end of this assignment precisely how extension to W_1 would be accomplished in general.

As a final note before we get started, some of this material may be considered as inspired from Chapter 12 section 5 of Richard Haberman's book *Elementary Applied Partial Differential Equations.* Haberman points out in that section that the problem he considers, which is a kind of special case of the problem (2) considered in this assignment with homogeneous zero order boundary conditions and noticeably no first order condition on the boundary, can be treated/solved using separation of variables and a Fourier expansion. As Haberman mentions, this approach was considered in his Chapter 2, and it is a pretty straightforward application of the same separation of variables technique we talked about for the heat equation and Laplace's equation. Notice, however, that if you really believe in this sort of thing as a model of the transverse oscillations of a vibrating string, then it is natural to include a first order boundary condition corresponding to changing (or forcing) the angle at which the string meets the zero boundary value, that is a condition like $u_x(a,t) = \sigma_0^-(t)$ and/or $u_x(b,t) = \sigma_0^+(t)$.

The problem(s) considered here are also related to Problem 4 of Assignment 7 which was given (by me) in MATH 6702 (Mathematical Methods in the Physical Sciences) in the Spring of 2023. Most of the techniques in the later parts of the development below were discussed in the lectures for that course, though the earlier parts were not discussed too much in the lectures, and I suspect not many of the students were able to, shall we say, take full advantage of the problem.

The overall organization of the development below is roughly according to the following outline:

0 Preliminaries

I Method of Characteristics

- **A** First Extension (second PDE $v = u_t - u_x$)
- **B** Second Extension (second PDE $v = u_t + u_x$)

II Divergence Theorem Method

- **A** The Method
- **B** Regularity and Weak Solutions

III Extension of Cauchy Data (reflection method)

- **A** The Method
- **B** Regularity and Propagation Speed

IV Applications

- **A** Tent Wave
- **B** Problem 4 of Assignment 7 (MATH 6702 Spring 2023)

Were I to consider (which I am now going to do) applying the method of characteristics to obtain an extended solution u of the wave equation on the domain W^-, then I would have a choice of how to factor the wave operator $\Box u = u_{tt} - u_{xx}$ and this will effect at least nominally the data that is propagated into the domain W^-. What I will call the "First Extension" in the outline above corresponds to setting

$$v = u_t - u_x.$$

Then we note that $v_t + v_x = 0$ because $v_t + v_x = \Box u$. Thus, we have two first order PDEs. The first PDE we consider is $v_t + v_x = 0$. This has characteristic field $(1, 1)$, and is set up to propagate our new boundary Cauchy data

$$\begin{cases} u(a,t) &= u_0^-(t), \quad 0 < t < b-a \\ u_x(a,t) &= \sigma_0^-(t), \quad 0 < t < b-a \end{cases}$$

off the vertical line $x = a$. This data propagates along a characteristic starting at some point (a, t_0) as indicated in Figure 2 (center).

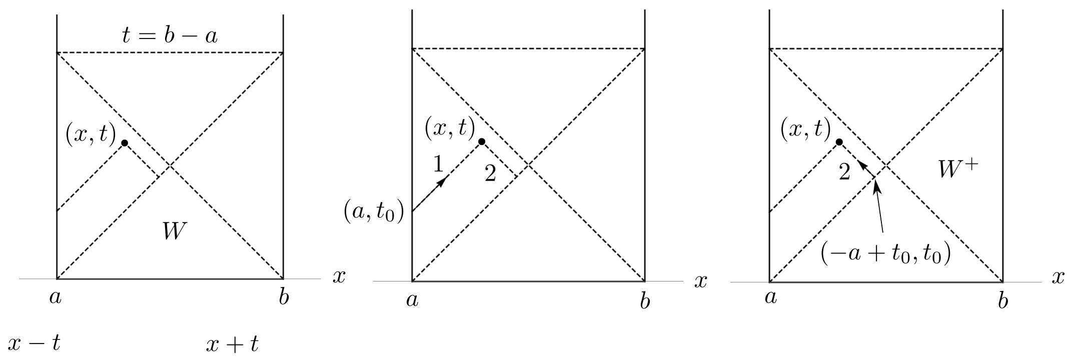

Figure 2: Method of characteristics to extend d'Alembert's solution to W^-. The numerals 1 and 2 have a double meaning in this figure. First there are two choices of the order of factoring the wave operator. If I factor $\Box u = v_t + v_x$ with $v = u_t - u_x$, then the **first PDE** I consider is $v_t + v_x = 0$, and to solve it I use a characteristic like the one with the numeral 1 next to it in the center illustration. Were I to factor in the reverse direction writing $\Box u = w_t - w_x$ with $w = u_t + u_x$, then my first equation would be $w_t - w_x = 0$, and I would propagate data from the d'Alembert solution using the backward propagating characteristic with the numeral 2 next to it in the center illustration. Thus, for the First Extension, I will find v first and then obtain u using the **second PDE** $u_t - u_x = v$ by propagating Cauchy data from the d'Alembert solution using the characteristic with the numeral 2 next to it in the illustration on the right. Thus, the numeral 2 in the center illustration may be associated with the **Second Extension** involving second PDE $u_t + u_x = w$ while the 2 in the right illustration may be associated with solving the **second PDE** for the First Extension now under consideration. The idea of Problems 1 and 2 below is that you understand all this and dance it around appropriately (and skillfully) to find a first extension of the d'Alembert solution to the domain W^+.

The forward propagating characteristic starting at (a, t_0) can be parameterized in the form

$$\gamma(t) = (a + (t - t_0), t) \qquad \text{for } t \geq t_0.$$

I keep in mind that the starting point (a, t_0) should be chosen in the end so that I can find the value of v at the point (x, t). This means eventually, I'll want to take

$$t_0 = a + t - x. \tag{3}$$

I'll also need to pick up the value of $v = u_t - u_x$ at the point (a, t_0) as a starting value. If I do this correctly, then integration will get me

$$v \circ \gamma(t) \equiv v(a, t_0).$$

Leaving out the details, I find

$$v(x, t) = (u_0^-)'(a + t - x) - \sigma_0^-(a + t - x). \tag{4}$$

Problem 1 (first extension to W^+, first equation) Factor the **wave operator** in such a way that you obtain and solve a homogeneous first order PDE on W^+ determined by Cauchy data

$$\begin{cases} u(b, t) &= u_0^+(t), \quad 0 < t < b - a \\ u_x(b, t) &= \sigma_0^+(t), \quad 0 < t < b - a \end{cases}$$

propagated backward in space along a characteristic starting at a point (b, t_0) on the vertical line $x = b$ in the boundary of the heat domain U.

Once I know the solution v given in (4), I am faced with a second PDE, namely

$$u_t - u_x = v.$$

The characteristic field for (the operator in) this PDE is also a constant field, but now it is backward propagating: $(-1, 1)$. Thus, in order to solve this PDE, I can introduce a characteristic starting at a point $(a + t_0, t_0)$ for some t_0 with $0 < t_0 < (b - a)/2$ and propagates backwards in space according to

$$\gamma(t) = (a + t_0 - (t - t_0), t) = (a + 2t_0 - t, t) \qquad \text{for } t \geq t_0$$

to the point $(x, t) \in W^-$. I can see now that the t_0 with which I should start is

$$t_0 = \frac{x + t - a}{2}.$$

Now, I have an inhomogeneous PDE, so I get an inhomogeneous ODE

$$\frac{d}{dt} u \circ \gamma = (u_0^-)'(2(t-t_0)) - \sigma_0^-(2(t-t_0)).$$

If I integrate this ODE correctly, and pick up an initial value $u(a+t_0, t_0)$ from the d'Alembert solution on $\partial W^- \cap \partial W$, then after some simplification (hopefully done and typed correctly) I find

$$\begin{aligned} u(x,t) = \frac{1}{2}[u_0(a) + u_0(x+t)] + \frac{1}{2}\int_a^{x+t} v_0(\xi)\, d\xi \\ + \frac{1}{2}[u_0^-(a+t-x) - u_0^-(0)] - \frac{1}{2}\int_0^{a+t-x} \sigma_0^-(\eta)\, d\eta. \end{aligned} \tag{5}$$

This is a pretty nice formula in some ways. One important property of this formula is that it does not depend on values of u_0 and v_0 outside the interval $[a, b]$. In other ways the formula (5) is not so great, or at least a little mysterious. We will get into the potentially troublesome aspects of (5) a little later. For now, let's see if you can fill in the details to find a similar formula for extending u to W^+.

Problem 2 (First extension to W^+) Let v denote the function you found in Problem 1. Solve the second PDE $u_t + u_x = v$ on W^+ to find an extension formula like the formula (5) I have given above for W^-.

You may have finished Problem 2 with some lack of confidence concerning the solution you have found, and if you did it correctly we'll see that you have good reason to be worried. On the other hand, there is a special case we can consider which should suggest we are on the right track. It also involves something that is good to keep in mind.

Here is the basic idea: If there is a way to extend the initial/Cauchy data u_0, v_0 to all of $\mathbb{R}$ so that, for example, the extensions satisfy $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$, then we should be able to extract from such an extension data on the vertical lines $x = a$ (my case/extension) and $x = b$ (your case/extension) for which the formula we found agrees with the d'Alembert solution arising from the extensions. I call this the **basic check**. It is a tiny bit subtle, but I'll indicate in some detail how it goes, then I'll leave it to you to apply the basic check to your extension formula from Problem 2.

I start by assuming u_0 and v_0 are actually defined on all of $\mathbb{R}$, so that d'Alembert's formula is globally applicable. This means I can find some special data on $x = a$ from the d'Alembert formula, namely,

$$u_0^-(t) = \frac{1}{2}[u_0(a-t) + u_0(a+t)] + \frac{1}{2}\int_{a-t}^{a+t} v_0(\xi)\, d\xi$$

and

$$\sigma_0^-(t) = \frac{1}{2}[u_0'(a-t) + u_0'(a+t)] + \frac{1}{2}[v_0(a+t) - v_0(a-t)].$$

Notice the value $u_0^-(t)$ is obtained simply by plugging $x = a$ into the d'Alembert formula (1) and the value $\sigma_0^-(t)$ is gotten by differentiating the d'Alembert formula with respect to x and than plugging $x = a$ into the result. Now, if I take these values in my formula (5), then what I should get is exactly the d'Alembert formula back. And that's what happens with my formula.

Problem 3 (the basic check) Apply the basic check to the extension formula you found in Problem 2.

(a) Write down the Cauchy data on $x = b$ corresponding to the solution determined by the formula (1) considered globally valid.

(b) Substitute the functions of t you found in part **(a)** into your formula from Problem 2 to get an expression depending only on u_0 and v_0.

(c) Show the expression you get in part **(b)** is exactly the d'Alembert formula.

If you can't do this, then you've probably made some kind of error. If it were me, I would want to find it.

Problem 4 (basic check for v) If you got through Problem 3 above, then you probably have a (nominally) correct formula for the extension on W^+. But let's imagine for a moment that you are having trouble completing the basic check, and you think there may even be a problem with the function you found in Problem 1. Devise a basic check for the function v you found in Problem 1 based on the same principle(s) used in Problem 3 and check that your solution of Problem 1 passes this basic check.

Another nice property of my solution (5) is that this function u defined on W^- extends continously to $\partial W^- \cap \partial W$ and agrees with the d'Alembert solution (extended from W) across the common boundary. This is called C^0 compatibility. To check this in my case, I consider the points $(x,t) = (a+t, t)$ and plug $x = a+t$ into my formula (5) and the d'Alembert formula (1) to make sure I get the same values.

Problem 5 (C^0 compatibility) Check the C^0 compatibility of your extension from Problem 2 with the d'Alembert formula along the common boundary $\partial W \cap \partial W^+$.

C^1 compatibility does not always hold, and in fact we should not expect it to always hold, and we should not want it to always hold. Problem 4 from Assignment 7 from MATH 6702 is one example where it cannot hold, and you should understand why after doing the next problem. I'll note that this behavior is also somewhat reminiscent of what we saw for the heat evolution when we imposed upon it a discontinuous forcing. That resulted in a singularity propagating through the flow for all times and the necessity of a weak solution. But there is a big difference here: That singularity was actively maintained by singular forcing at every positive time. This had to be the case because the heat operator is smoothing. Any initial singularity gets immediately smoothed out. But the wave operator is **not smoothing** and any initial singularity tends to naturally get propagated in the wave motion/evolution.

To check C^1 compatibility, I again evaluate along my boundary curve $\partial W^- \cap \partial W$. First I evaluate

$$u_x\big|_{W^-}(a+t,t)$$

using my solution (5). Then I evaluate

$$u_x\big|_{W}(a+t,t)$$

using (1). Setting the two equal, I find the interesting condition for C^1 compatibility:

$$\sigma_0^-(0) - (u_0^-)'(0) = u_0'(a) - v_0(a). \tag{6}$$

I can and should check the time derivatives as well evaluating first

$$u_t\big|_{W^-}(a+t,t)$$

using my solution (5) and then

$$u_t\big|_{W}(a+t,t)$$

using (1). I get the same condition, so I have a theorem.

Theorem 1 (C^1 compatibility for the first extension) The formula (5) gives a continuous extension of the d'Alembert formula to W^- which will be a C^1 extension if and only if (6) holds.

Problem 6 (C^1 compatibility for the first extension) Find a C^1 compatibility condition like (6) for your first extension obtained in Problem 2 above. Here are some other good things to do:

(a) Explain the "most natural" circumstances under which this condition holds.

(b) Describe some "more exotic" circumstances under which you get a C^1 extension.

The time has come to face a more serious shortcoming of the first extension formula(s). If I evalute my formula on $x = a$ I find

$$\begin{aligned} u(a,t) = \frac{1}{2}[u_0(a) + u_0(a+t)] + \frac{1}{2}\int_a^{a+t} v_0(\xi)\, d\xi \\ + \frac{1}{2}[u_0^-(t) - u_0^-(0)] - \frac{1}{2}\int_0^t \sigma_0^-(\eta)\, d\eta. \end{aligned} \tag{7}$$

If I set this equal to $u_0^-(t)$, I get the condition

$$\begin{aligned} u_0^-(t) = u_0(a) + u_0(a+t) + \int_a^{a+t} v_0(\xi)\, d\xi \\ - u_0^-(0) - \int_0^t \sigma_0^-(\eta)\, d\eta. \end{aligned} \tag{8}$$

After some reflection I realize I have no reason to believe this condition holds. That is, I've assigned what seemed to be perfectly good Cauchy data along the vertical line $x = a$. This gave me a continuous extension which I thought made sense and I understood. Of course, there was some single point condition (6) to ensure the extension was C^1, but this was not unexpected and made good sense. So I imposed seemingly perfectly good Cauchy data along $x = a$, but my solution method may ignore my zero order Cauchy data or at least fail to realize it. Since the integral condition on σ_0 in (8) is complicated, I will turn to the first order Cauchy data directly.

Differentiating my formula with respect to x and evaluating at $x = a$, I find

$$\begin{aligned} u_x(a,t) = \frac{1}{2}u_0'(a+t) + \frac{1}{2}v_0(a+t) \\ - \frac{1}{2}(u_0^-)'(t) + \frac{1}{2}\sigma_0^-(t). \end{aligned} \tag{9}$$

Setting this equal to my first order Cauchy data $\sigma_0^-(t)$ along $x = a$, I find the more tractable condition

$$\sigma_0^-(t) + (u_0^-)'(t) = u_0'(a+t) + v_0(a+t). \tag{10}$$

Since u_0' and v_0 are given functions, this is relatively easy to interpret: If I attempt to assign Cauchy data of zero order given by $u_0^-(t)$ along $x = a$, then the first order Cauchy data, or the value of $\sigma_0^-(t) = u_x(a,t)$ is not arbitrary but is entirely determined by the formula

$$\sigma_0^-(t) = u_0'(a+t) + v_0(a+t) - (u_0^-)'(t). \tag{11}$$

Problem 7 (compatibility at $x = b$) Apply the reasoning above to your first extension from Problem 2.

(a) Find a condition on $\sigma_0^+(t)$ along $x = b$ corresponding to the prescription (11).

(b) Assuming your condition from part **(a)** prescribes $\sigma_0^+(t)$, does your extension formula give a function satisfying the zero order Cauchy condition

$$u(b,t) = u_0^+(t)$$

along $x = b$? If not, what additional compatibility condition is required/obtained?

Obviously, the second extension is obtained by using the reverse factoring of the operator, and a series of checks and compatibility conditions should follow up finding the formula. Since I've already composed seven problems above, I'm going to simply state the formula and leave that out of the problems: If you take the second PDE as $u_t + u_x = w$ on W^- with $w_t - w_x = 0$, then you find

$$w(x,t) = u_0'(x+t) + v_0(x+t)$$

and

$$\tilde{u}(x,t) = u_0^-(0) + \frac{1}{2}[u_0(x+t) - u_0(2a+t-x)] + \frac{1}{2}\int_{2a+t-x}^{x+t} v_0(\xi)\, d\xi.$$

Again, there are various basic checks one can do with these extensions. Then there are interesting properties and compatibility conditions, as well as conditions under which this is the same extension obtained in (5). I have not included these questions in the problems.

What I'm going to do next is rather simple, and I hope you'll like it. Again, say I have a point $(x, y) \in W^-$ and I wish to find the extension of the d'Alembert solution on W extending continuously to W^- with some prescribed zero order Cauchy data $u_0^-(t)$ prescribed on the boundary $x = a$. I consider a rectangle

$$R = \big\{(\xi, \tau) \in W^- : t_0 + |\xi - t_0| < \tau < t - |\xi - t|\big\}$$

where $(a+t_0, t_0)$ is a corner point we have not previously considered. Such a rectangle is illustrated on the left in Figure 3. We have considered two sides of this rectangle

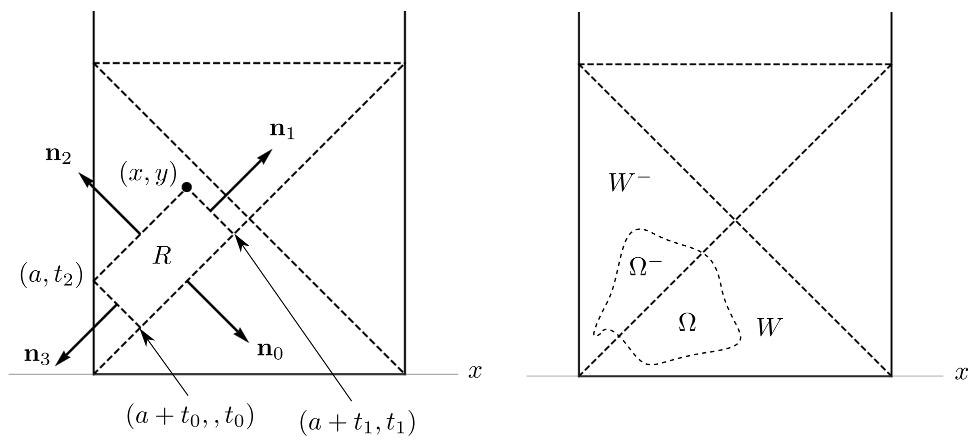

Figure 3: Integration and the wave equation. A rectangular domain bounded by characteristic curves (left). The support of a test function crossing the boundary between W^- and W (right).

previously, but I've changed the notation slightly because we will be integrating on the entire rectangle and its boundary. The two parameterizations we have considered now take the forms

$$\gamma_1(\tau) = (a + 2t_1 - \tau, \tau) \qquad \text{for } t_1 \le \tau \le t \qquad \text{with} \qquad t_1 = \frac{x + t - a}{2}$$

and

$$\gamma_2(\tau) = (a + \tau - t_2, \tau) \qquad \text{for } t_2 \le \tau \le t \qquad \text{with} \qquad t_2 = a + t - x.$$

One new path runs along $\partial W^- \cap \partial W$ from $(a+t_0, t_0)$ to $(a+t_1, t_1)$, that is

$$\gamma_0(\tau) = (a+\tau, \tau) \qquad \text{for } t_0 \le \tau \le t_1 \text{with} \qquad t_0 = \frac{a+t-x}{2}.$$

and and t_1 is given above. The value of t_0 satisfies $t_1 - t_0 = t - t_2$. Finally, the last new path is given by

$$\gamma_3(\tau) = (a-(t-\tau), \tau) \qquad \text{for } t_0 \le \tau \le t_2.$$

Let's use the symbol ∇ to denote the full x, t gradient

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right)$$

with associated space-time divergence $\mathbb{D}\text{iv}$ so that

$$\Box u = -\mathbb{D}\text{iv}(u_x, -u_t).$$

Thus, if u solves $\Box u = 0$, we can integrate over R and apply the divergence theorem to see

$$0 = \int_R \Box u = -\int_{\partial R} (u_x, -u_t) \cdot \mathbf{n} \tag{12}$$

where $\mathbf{n}$ is the outward unit normal to ∂R.

Now I will write out the integral over the side Γ_1 of ∂R with outward normal $\mathbb{N}_1 = (1, 1)$:

$$\begin{aligned} \int_{\Gamma_1} (u_x, -u_t) \cdot \mathbf{n}_1 &= \int_{t_1}^{t} [u_x(a+2t_1-\tau, \tau) - u_t(a+2t_1-\tau, \tau)]\, d\tau \\ &= -\int_{t_1}^{t} \frac{d}{d\tau} u(a+2t_1-\tau, \tau)\, d\tau \\ &= -u(x,t) + u(a+t_1, t_1). \end{aligned}$$

Notice one term here involves $u(x, t)$ which is the quantity we are looking to find for an extension. The other term involves $u(a + t_1, t_1)$ which we can consider given by the d'Alembert formula on W.

If you write out the integrals over all four sides of R similar things happen and you see the divergence theorem applied in (12) is an equation for the desired quantity $u(x, y)$ in terms of values at the corners of R which are all completely known. Thus, you can get an extension formula this way.

Problem 8 (divergence theorem) Apply the divergence theorem to an appropriate rectangle in W^+ to obtain an extension formula for a solution of the wave equation on W^+.

(a) Does the extension formula you find match the extension formula obtained above by factoring the operator and applying the method of characteristics?

(b) Are there associated compatibility conditions?

We have noted that we cannot always expect an extended solution of the wave equation to be C^1 across the boundary between W^- and W (or your boundary between W and W^+).

We say a continuous function $u \in C^0(U_1)$ satisfies $\Box u = 0$ weakly on the truncated wave domain U_1 if

$$\int u(\phi_{tt} - \phi_{xx}) = 0 \qquad \text{for every } \phi \in C_c^\infty(U_1). \tag{13}$$

Let me imagine for a moment that u is a C^2 solution in W^- which extends the d'Alembert solution on W continuously. Now say I have a test function ϕ with support $\Omega^- \cup \Omega$ crossing $\partial W^- \cap \partial W$ as indicated on the right in Figure 3. Notice I can express the integral relation in the weak solution condition (13) as

$$\int_{\Omega^-} u \mathbb{D}\text{iv}(\phi_x, -\phi_t) + \int_\Omega u \mathbb{D}\text{iv}(\phi_x, -\phi_t) = 0.$$

Thus, using the product formula for the divergence

$$\mathbb{D}\text{iv}[u(\phi_x, -\phi_t)] = \nabla u \cdot (\phi_x, -\phi_t) + u\mathbb{D}\text{iv}(\phi_x, -\phi_t)$$

on each subdomain I obtain

$$\int_{\partial\Omega^-} u(\phi_x, -\phi_t) \cdot \mathbb{N}^- - \int_{\Omega^-} (u_x, -u_t) \cdot \nabla\phi + \int_{\partial\Omega} u(\phi_x, -\phi_t) \cdot \mathbb{N} - \int_\Omega (u_x, -u_t) \cdot \nabla\phi = 0$$

where $\mathbb{N}^-$ is the unit outward normal to $\partial\Omega^-$ and $\mathbb{N}$ is the unit outward normal to $\partial\Omega$. Now consider the integrals in the sum

$$\int_{\partial\Omega^-} u(\phi_x, -\phi_t) \cdot \mathbb{N}^- + \int_{\partial\Omega} u(\phi_x, -\phi_t) \cdot \mathbb{N}. \tag{14}$$

Since ϕ has compact support $\Omega^- \cup \Omega$, we know ϕ and all its derivatives vanish on the portions of $\partial\Omega^-$ and $\partial\Omega$ outside $\partial W^- \cap \partial W$. That is, the quantity in (14) is given by

$$\int_{\partial\Omega^- \cap \partial W} u(\phi_x, -\phi_t) \cdot \mathbb{N}^- + \int_{\partial\Omega \cap \partial W} u(\phi_x, -\phi_t) \cdot \mathbb{N}.$$

But on this portion ϕ is smooth, u is continuous, and $\mathbb{N}^- = -\mathbb{N}$. We conclude that the quantity in (14) is zero, and

$$\int_{\Omega^-} (u_x, -u_t) \cdot \nabla\phi + \int_{\Omega} (u_x, -u_t) \cdot \nabla\phi = 0$$

Now, using the product formula

$$\begin{aligned} \mathbb{D}\text{iv}[\phi(u_x, -u_t)] &= \nabla\phi \cdot (u_x, -u_t) + \phi \mathbb{D}\text{iv}(u_x, -u_t) \\ &= \nabla\phi \cdot (u_x, -u_t) - \phi \Box u \end{aligned}$$

my identity further becomes

$$\int_{\partial\Omega^-} [(u_x, -u_t) \cdot \mathbb{N}^-]\phi + \int_{\partial\Omega} [(u_x, -u_t) \cdot \mathbb{N}]\phi = 0.$$

Again, shedding the portions of boundary where ϕ vanishes and using the relation $\mathbb{N}^- = \mathbb{N}$ on the remainder we conclude

$$\int_{\partial\Omega \cap \partial W} \left[\left(u_x\big|_W, -u_t\big|_W \right) - \left(u_x\big|_{W^-}, -u_t\big|_{W^-} \right) \right] \cdot \mathbb{N}\ \phi = 0.$$

Finally, since $\mathbb{N} = (-1, 1)$ we conclude from the fundamental lemma of vanishing integrals that

$$(u_x - u_t)\big|_W (a + t, t) = (u_x - u_t)\big|_{W^-} (a + t, t) \tag{15}$$

for $0 < t < (b - a)/2$ is a necessary condition for a C^0 weak solution of the wave equation along $\partial W^- \cap \partial W$.

The derivatives in this condition can now be written in terms of the extension formulas given above (on the W^- side) and the d'Alembert formula (on the W side).

Problem 9 (Interior C^0 weak solutions of the 1-D wave equation) Find the corresponding necessary condition for a C^0 weak solution across the line segment $\partial W \cap \partial W^+$, and determine if your extension formulas from Problems 2 and 8 satisfy this condition.

Part III Extension of the Cauchy data u_0 and v_0

If I assume u_0 and v_0 are extended so that the boundary value $u_0^-(t)$ is given by the global d'Alembert solution along $x = a$ with $0 < t < b - a$, that is

$$\frac{1}{2}[u_0(a-t) + u_0(a+t)] + \frac{1}{2}\int_{a-t}^{x+t} v_0(\xi)\, d\xi = u_0^-(t)$$

and I also assume the compatibility condition (11) and the continuity condition

$$u_0(a) = u_0^-(0)$$

so that the zero order Cauchy data is achieved by my extension (5), then I also have

$$\begin{aligned} u_0^-(t) = \frac{1}{2}[u_0(a) + u_0(a+t)] + \frac{1}{2}\int_a^{a+t} v_0(\xi)\, d\xi \\ + \frac{1}{2}[u_0^-(t) - u_0^-(0)] - \frac{1}{2}\int_0^t \sigma_0^-(\eta)\, d\eta. \end{aligned} \tag{16}$$

Equating these expressions and simplifying, I obtain a condition on the extension:

$$u_0(x-t) + \int_{x-t}^0 v_0(\xi)\, d\xi = -u_0(-(x-t)) - \int_0^{-(x-t)} v_0(\xi)\, d\xi.$$

Again, this is nominally a little complicated but somewhat suggestive if you think about it. In particular, differentiating we get

$$u_0'(x-t) - v_0(x-t) = u_0'(-(x-t)) + v_0(-(x-t)).$$

These formulas should suggest to you an obvious choice for the extension so that the solution on the truncated wave domain for a finite interval is given by a global d'Alembert solution, at least in some special cases.

Problem 10 (extension/reflection) Apply the discussion above to the special case in which $a = 0$, $b = \ell > 0$, $u_0^-(t) \equiv 0$, $u_0^+ \equiv 0$, $v_0(x) \equiv 0$ and

$$u_0(x) = c\frac{\ell}{2} - c\left|x - \frac{\ell}{2}\right|$$

for some $c > 0$ and $0 < x < \ell$. What do you find for the extension of u_0 and v_0?