MATH 4581 Lecture 18 Tuesday Oct. 26, 2021

Maximum Principle:

bounded open connected subset of $\mathbb{R}^n$

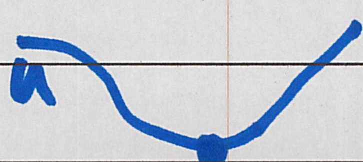
$u \in C^0(\bar{\mathcal{U}})$

If $\Delta u = 0$ on $\mathcal{U}$, then

$$u(\mathbf{x}) < \max_{\mathbf{p} \in \partial \mathcal{U}} u(\mathbf{p}) \quad \text{for } \mathbf{x} \in \mathcal{U}$$

unless $u \equiv$ constant.

Can u attain a minimum value?



Say $u(\mathbf{x}_0) = \min_{\mathbf{p} \in \bar{\mathcal{U}}} u(\mathbf{p})$

$\cdot\, \mathbf{x}_0 \in \mathcal{U}$

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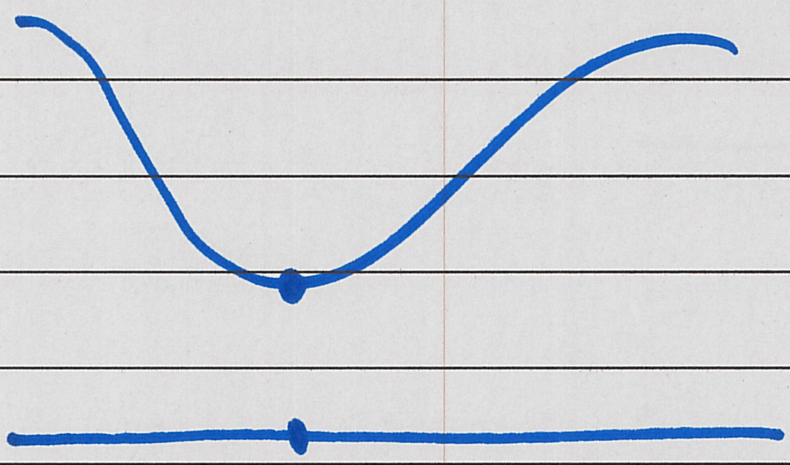


$x_0 \in \mathcal{U}$

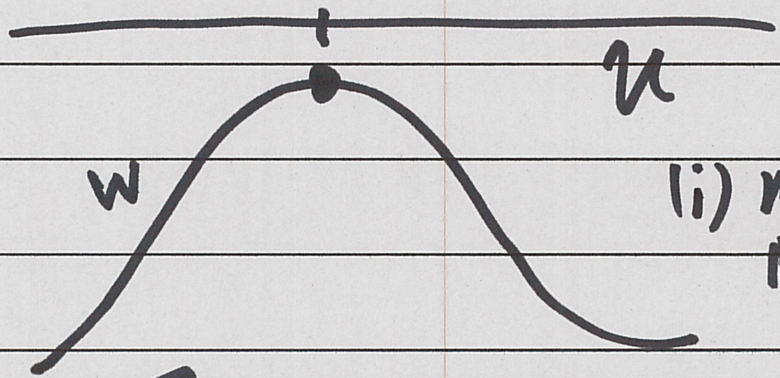
$u(x_0) = \min_{P \in \overline{\mathcal{U}}} u(P).$

$u(x_0) \leq u(P)$ for all P.

$w(x_0) \geq w(P)$ for all P.

Consider $w : \mathcal{U} \to \mathbb{R}$ by $w(x) := -u(x).$

$\Delta w = -\Delta u = 0.$



Then $w(x_0)$

(i) $\max_{P \in \overline{\mathcal{U}}} w(P) = \max_{P \in \partial \mathcal{U}} w(P)$

(ii) $w(x) < \max w(P)$, $x \in \mathcal{U}$ unless $w \equiv c$.

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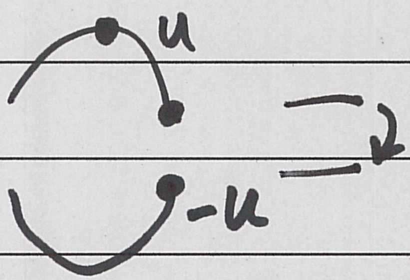
To <u>show</u> $\min_{p \in \overline{U}} u(p) = \min_{p \in \partial U} u(p)$.

Assume by way of contradiction that



$$\underbrace{\min_{p \in \overline{U}}}_{U \cup \partial U} u(p) < \min_{p \in \partial U} u(p)$$

Then there is some $x_0 \in U$ with $u(x_0) < \min_{p \in \overline{U}} u(p)$

This means $w(x_0) > \max_{p \in \partial U} w(p)$.

$$-u(x_0) > \max_{p \in \partial U} [-u(p)] = -\overset{\min}{\cancel{\max}}_{p \in \partial U} u(p)$$

[redacted]

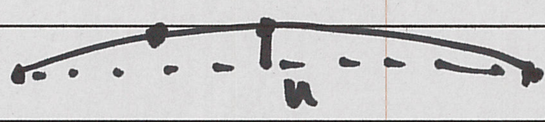
$\leftarrow \sigma$ = wave speed.

$$\begin{cases} u_{tt} = \sigma^2 u_{xx} \\ \boxed{u(0,t) = 0 = u(L,t)} \\ u(x,0) = u_0(x) \\ \boxed{u_t(x,0) = v_0(x)} \end{cases}$$

1-D wave equation

boundary values

initial position and velocity



Model for stringed instruments.

$u = A(x)B(t)$

$AB'' = \sigma^2 A'' B$

$$\frac{A''}{A} = \frac{1}{\sigma^2}\frac{B''}{B} = \lambda$$

$A(0) = 0 = A(L)$

[redacted]

$$A_j = \cancel{\alpha_j} \sin \frac{j\pi x}{L}$$ (Sturm-Liouville problem)

$$\lambda = -\frac{j^2\pi^2}{L^2}$$

$$B_j'' = -\frac{j^2\pi^2}{L^2}\sigma^2 B$$

$$\boxed{B_j = a_j \cos\left(\frac{j\pi\sigma}{L}t\right) + b_j \sin\left(\frac{j\pi\sigma}{L}t\right)}$$

$$u_j = \sin\frac{j\pi x}{L}\left[a_j \cos\left(\frac{j\pi\sigma}{L}t\right) + b_j \sin\left(\frac{j\pi\sigma}{L}t\right)\right]$$

$$u = \sum_{j=1}^{\infty} \sin\frac{j\pi x}{L}\left[a_j \cos\left(\frac{j\pi\sigma}{L}t\right) + b_j \sin\left(\frac{j\pi\sigma}{L}t\right)\right].$$

Initial position:

$$u(x,0) = \sum_{j=1}^{\infty} a_j \sin\frac{j\pi x}{L} = u_0(x)$$

[redacted]

<u>initial velocity:</u>

$$u_t = \sum_{j=1}^{\infty} \sin\left(\frac{j\pi x}{L}\right)\left[-\frac{j\pi\sigma}{L} a_j \sin\left(\frac{j\pi\sigma}{L}t\right) + \underline{\underline{\frac{j\pi\sigma}{L}}} b_j \cos\left(\frac{j\pi\sigma}{L}t\right)\right]$$

$$u_t(x,0) = \sum_{j=1}^{\infty} \underbrace{\frac{j\pi\sigma}{L} b_j}_{=\beta_j} \sin\left(\frac{j\pi x}{L}\right) = V_0(x)$$

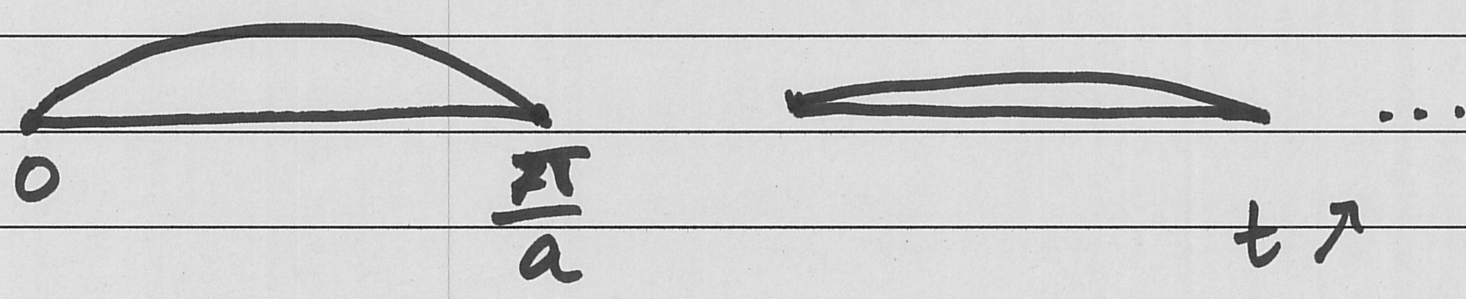
$\beta_j \leftarrow$ Fourier sine coefficients for V_0

$$b_j = \frac{L}{j\pi\sigma}\beta_j \qquad j = 1, 2, 3, \ldots$$

[redacted]

typical solution of the heat equation:

$$\boxed{u(x,t) = e^{-a^2 t} \sin(ax)}$$



typical solution of the wave equation

$$u(x,t) = \sin(\omega t)\sin(ax)$$

$$(\underset{\substack{\uparrow \\ \text{frequency}}}{\omega} = \underset{\substack{\nwarrow \\ \text{wave speed.}}}{\sigma}\, a)$$

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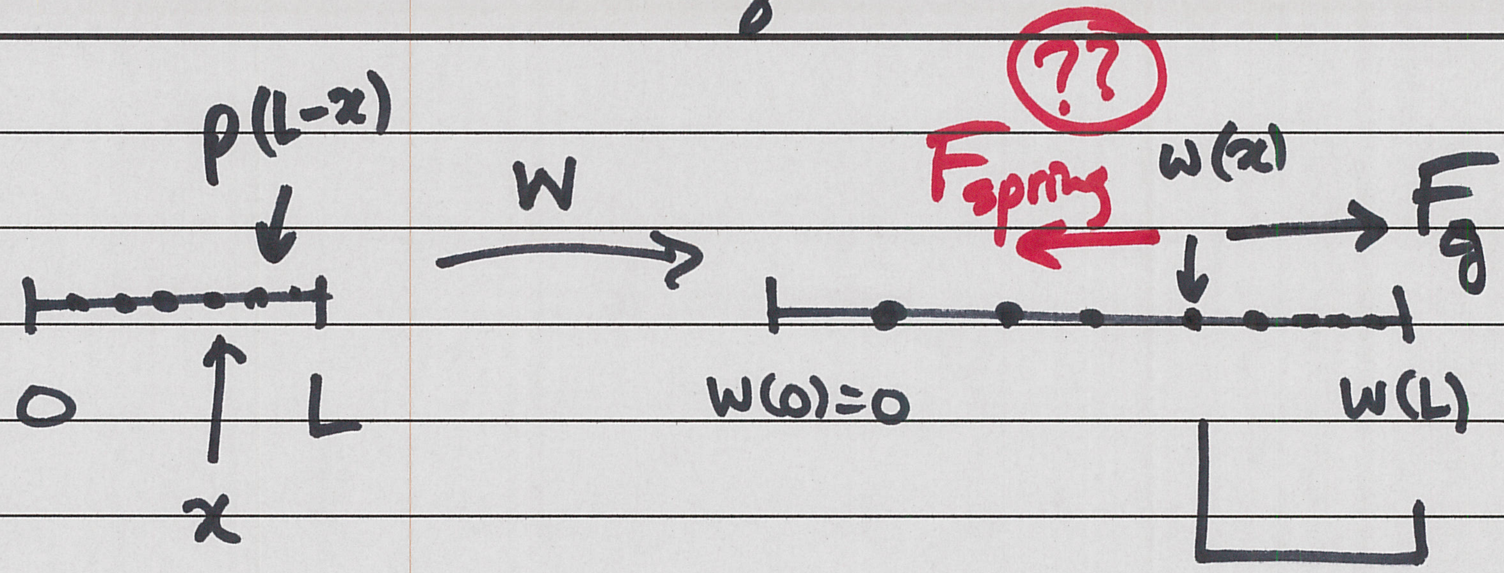

mass density ρ.

$\rho(L-x)$

$$F_g = g\rho(L-x)$$

$$F_{spring} = -\epsilon\left(\frac{\ell'}{\ell} - 1\right)$$

[redacted]