MATH 4581 Lecture 25 Thursday November 18, 2021

LAST TIME : Sturm-Liouville Theory

$$\begin{cases} (py')' + (q+\lambda r)y = 0 \\ \quad + \text{ boundary values} \end{cases}$$

→ get sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_j \nearrow \infty$

→ eigenfunctions $y_1, y_2, y_3, \ldots$

orthogonality: $\int y_i y_j r$

[redacted]

$$\begin{cases} \Delta u = -\lambda u & \text{on } R = [0,L]\times[0,M] \\ u\big|_{\partial R} = 0 \end{cases}$$

$$u_{jk} = \sin\left(\frac{j\pi}{L}x\right)\sin\left(\frac{k\pi}{M}y\right)$$

$$\lambda_{jk} = \frac{j^2\pi^2}{L^2} + \frac{k^2\pi^2}{M^2}$$

[redacted]

$$\begin{cases} \Delta B = -\lambda B & \text{on } B_a(0) \\ B\big|_{\partial B_a(0)} = 0 \end{cases}$$

$$\alpha(r)\beta(\theta) = B(r\cos\theta, r\sin\theta)$$

$$\alpha'\beta = B_x \cos\theta + B_y \sin\theta$$

$$\alpha\beta' = -r\sin\theta\, B_x + r\cos\theta\, B_y$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \alpha'\beta \\ \alpha\beta' \end{pmatrix}$$

[redacted]

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha'\beta \\ \alpha\beta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\, \alpha'\beta - \frac{1}{r}\sin\theta\, \alpha\beta' \\ \sin\theta\, \alpha'\beta + \frac{1}{r}\cos\theta\, \alpha\beta' \end{pmatrix}$$

$$\begin{cases} \alpha''\beta = B_{xx}\cos^2\theta + 2B_{xy}\cos\theta\sin\theta + B_{yy}\sin^2\theta \\ \alpha\beta'' = r^2\sin^2\theta\, B_{xx} - 2r^2\cos\theta\sin\theta\, B_{xy} + r^2\cos^2\theta\, B_{yy} \\ \qquad - r\cos\theta\, B_x - r\sin\theta\, B_y \end{cases}$$

$$\alpha'\beta = B_x\cos\theta + B_y\sin\theta$$

$$\alpha\beta' = -r\sin\underline{\theta}\, B_x + r\cos\underline{\theta}\, B_y$$

[redacted]

$$\alpha''\beta = \cos^2\theta\, B_{xx} + 2\cos\theta\sin\theta\, B_{xy} + \sin^2\theta\, B_{yy}$$

$$\alpha\beta'' = r^2\sin^2\theta\, B_{xx} - 2r^2\cos\theta\sin\theta\, B_{xy} + r^2\cos^2\theta\, B_{yy}$$
$$-r\cos\theta\, B_x - r\sin\theta\, B_y$$

$$\alpha''\beta + \frac{1}{r^2}\alpha\beta'' = B_{xx} + B_{yy} - r\cos\theta\, B_x - r\sin\theta\, B_y$$
$$= -\lambda\alpha\beta - r\alpha'\beta$$

$$\begin{cases} \alpha''\beta + \frac{1}{r^2}\alpha\beta'' + \frac{1}{r}\alpha'\beta + \lambda\alpha\beta = 0 \\ \alpha(a)\beta(\theta) = 0 \end{cases}$$

$\alpha(0)\beta(\theta)$ well-defined (finite)

[redacted]

$$\alpha''\beta + \frac{1}{r^2}\alpha\beta'' + \frac{1}{r}\alpha'\beta + \lambda\alpha\beta = 0$$

$$\frac{\alpha''}{\alpha} + \frac{1}{r^2}\frac{\beta''}{\beta} + \frac{1}{r}\frac{\alpha'}{\alpha} + \lambda = 0$$

$$r^2\frac{\alpha''}{\alpha} + r\frac{\alpha'}{\alpha} + \lambda r^2 = -\frac{\beta''}{\beta} = \mu$$

second separation constant

GOOD PROBLEM:

$$\begin{cases} \beta'' = -\mu\beta \\ \beta(0) = \beta(2\pi) \\ \beta'(0) = \beta'(2\pi) \end{cases}$$

$$\boxed{\beta_j = a_j \cos j\theta + b_j \sin j\theta, \; \mu_j = j^2}$$

[redacted]

Radial Problem:

$$r^2 \frac{\alpha''}{\alpha} + r\frac{\alpha'}{\alpha} + \lambda r^2 = +j^2$$

$$r(r\alpha')' + (-j^2 + \lambda r^2)\alpha = 0$$

Sturm-Liouville form: $(r\alpha')' + \left(-\frac{j^2}{r} + \lambda r\right)\alpha = 0$

$$\begin{cases} r(r\alpha')' + (-j^2+\lambda r^2)\alpha = 0 \\ \alpha(a) = 0 \\ \alpha(0) \in \mathbb{R} \end{cases}$$

compare to $\begin{cases} y'' + \lambda y = 0 \\ y(0)=0,\ y(L)=0 \end{cases}$

[redacted]

This is <u>not</u> a regular Sturm-Liouville [redacted] problem, but it is a "regular singular" [redacted] problem.

Singular Sturm-Liouville Problem

$$\begin{cases} r(r\alpha')' + (-j^2 + \lambda r^2)\alpha = 0 \\ \alpha(a) = 0, \quad \alpha(0) \in \mathbb{R} \end{cases}$$

Expect a sequence of eigenvalues/eigenfunctions.

First task: Understand the (non-constant coeff) ODE.

new A → $A(\xi) = \alpha\left(\frac{\xi}{\sqrt{\lambda}}\right)$; $r = \frac{\xi}{\sqrt{\lambda}}$

$A' = \frac{1}{\sqrt{\lambda}}\alpha'\left(\frac{\xi}{\sqrt{\lambda}}\right)$; $A'' = \frac{1}{\lambda}\alpha''\left(\frac{\xi}{\sqrt{\lambda}}\right)$

ODE: $r\alpha'' + \alpha' + \left(-\frac{j^2}{r} + \lambda r\right)\alpha = 0$ [redacted]

$$\frac{\xi}{\sqrt{\lambda}} \cdot \lambda A'' + \sqrt{\lambda}A' + \left(-\frac{j^2\sqrt{\lambda}}{\xi} + \lambda\frac{\xi}{\sqrt{\lambda}}\right)A = 0$$

$$\boxed{\xi^2 A'' + \xi A' + (\xi^2 - j^2) A = 0}$$

Bessel's ODE of order $\overset{\nu}{j}$

Write as:

$$x^2 y'' + x y' + (x^2 - \overset{\nu^2}{j^2}) y = 0$$

power series:
$$y = \sum a_j x^j$$

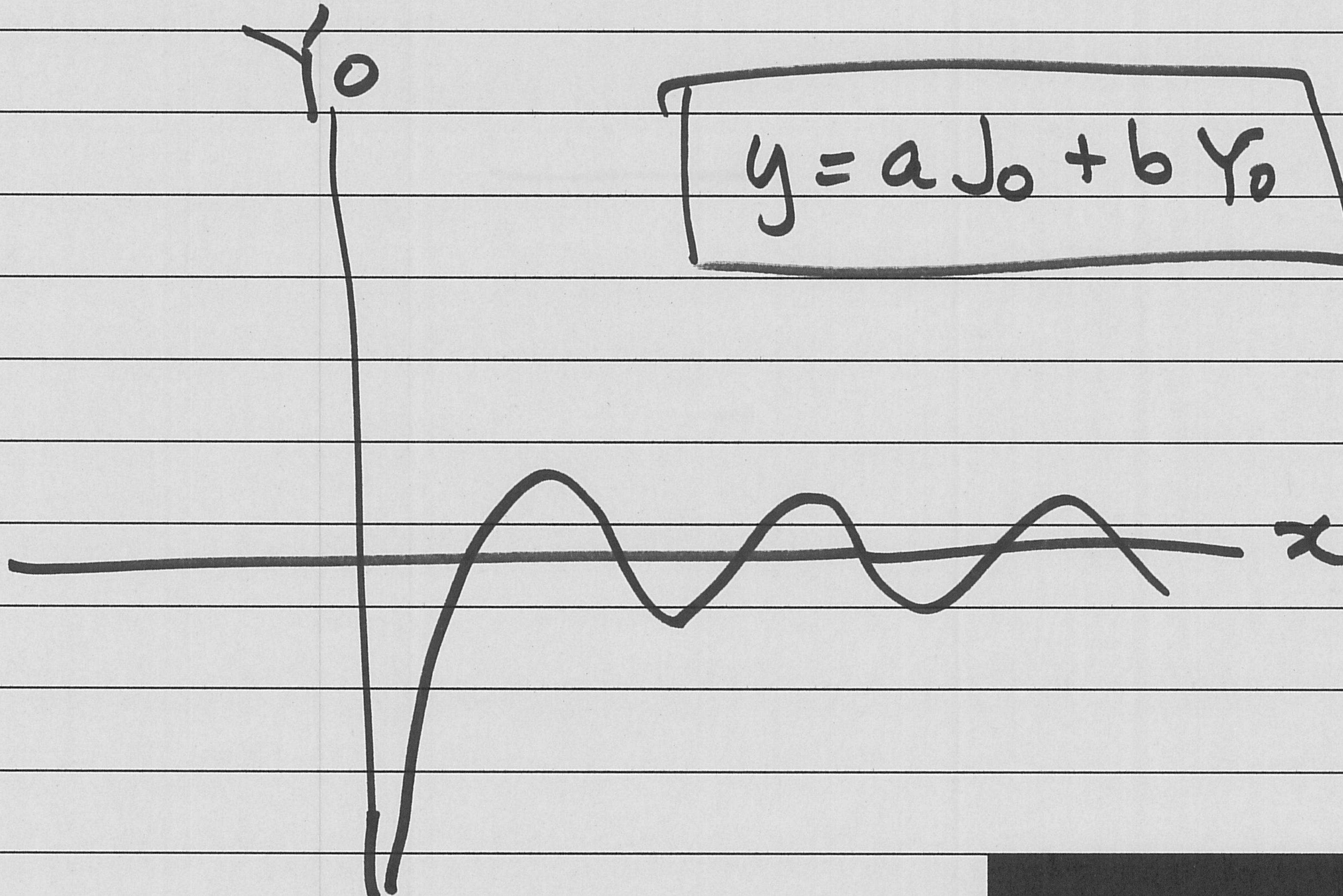
Method of Froebenius:

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+\gamma}$$

$$y'(x) = \sum_{j=0}^{\infty} (j+\gamma) a_j x^{j+\gamma-1}$$

$$y''(x) = \sum_{j=0}^{\infty} (j+\gamma)(j+\gamma-1) a_j x^{j+\gamma-2}$$ [redacted]

$$\boxed{y = a J_0 + b Y_0}$$



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