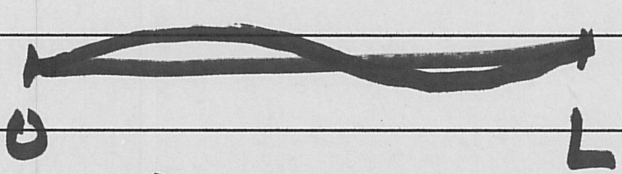
MATH 4581 Lecture 26 November 23, 2021

Review

1-D $\quad u_{tt} = \sigma^2 u_{xx}, \quad u = A(x)B(t)$

$$AB'' = \sigma^2 A'' B$$

$$\frac{B''}{\sigma^2 B} = \frac{A''}{A} = -\frac{j^2\pi^2}{L^2}, \quad j = 1, 2, 3, \ldots$$



$$A_j = \sin\left(\frac{j\pi}{L}x\right)$$

$$u_j = \left[a_j \cos\left(\frac{\sigma j\pi}{L}t\right) + b_j \sin\left(\frac{\sigma j\pi}{L}t\right)\right] \sin\left(\frac{j\pi}{L}x\right).$$

$$u_j = \left[a_j \cos\left(\frac{\sigma j \pi}{L} t\right) + b_j \sin\left(\frac{\sigma j \pi}{L} t\right) \right] \sin\left(\frac{j\pi}{L} x\right)$$

oscillation $B_j(t)$ waveform

$$\begin{cases} a_j = \text{initial displacement} \\ b_j \dfrac{\sigma j \pi}{L} = \text{initial velocity} \end{cases}$$

[redacted]

2-D rectangular drum

$$R = (0,L) \times (0,M)$$

$$\begin{cases} u_{tt} = \sigma^2 \Delta u & \text{on } R \times (0,\infty) \\ u\big|_{\partial R} = 0, & t > 0 \end{cases}$$

$$u = A(x)\,B(y)\,C(t)$$

$$ABC'' = \sigma^2 \left(A''BC + AB''C \right)$$

$$\frac{C''}{\sigma^2 C} = \underbrace{\frac{A''}{A} + \frac{B''}{B}} = \lambda$$

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$$\frac{C''}{\sigma^2 C} = \frac{A''}{A} + \frac{B''}{B} = \lambda$$

2^{nd} separation :

$$\frac{A''}{A} = -\frac{B''}{B} + \lambda = \underset{\mu_j}{\overset{\shortparallel}{\mu}} = -\frac{j^2\pi^2}{L^2}$$

$$A_j = \sin\left(\frac{j\pi}{L}x\right)$$

$$\begin{cases} B'' = \Big(\underbrace{\frac{j^2\pi^2}{L^2} + \lambda}\Big) B \\ B(0) = 0 = B(M) \end{cases} \leftarrow -\frac{k^2\pi^2}{M^2}$$

[redacted]

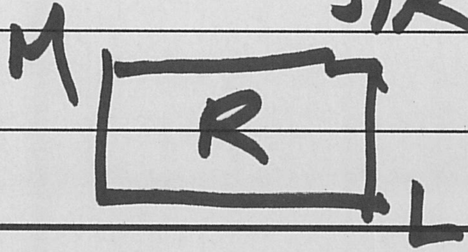
$$\begin{cases} B'' = \left(\dfrac{j^2\pi^2}{L^2} + \lambda \right) B \\ B(0) = 0 = B(M) \end{cases}$$

$$\boxed{\lambda_{jk} = -\left(\frac{k^2}{M^2} + \frac{j^2}{L^2}\right)\pi^2}$$

$$C_{jk}(t) = a_{jk} \cos\left(\sigma\left(\frac{j^2}{L^2} + \frac{k^2}{M^2}\right)\pi^2 t\right)$$
$$+ b_{jk} \sin\left(\sigma\left(\frac{j^2}{L^2} + \frac{k^2}{M^2}\right)\pi^2 t\right)$$

$\boxed{\text{oscillation}}$

$$u = \sum_{j,k} C_{jk} \sin\left(\frac{j\pi}{L} x\right) \sin\left(\frac{k\pi}{M} y\right)$$



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2-D on circular drum. $u_{tt} = \sigma^2 \Delta u$

$$U(r,\theta,t) = u(r\cos\theta, r\sin\theta, t)$$

$$U_{tt} = \sigma^2 \left[\frac{1}{r}(rU_r)_r + \frac{1}{r^2} U_{\theta\theta} \right]$$

← from Lecture 25.

$$U = A(r)B(\theta)C(t)$$

$$ABC'' = \sigma^2 \left[\frac{BC}{r}(rA')' + \frac{AC}{r^2} B'' \right]$$

$$\frac{C''}{\sigma^2 C} = \frac{(rA')'}{rA} + \frac{B''}{r^2 B} = -\lambda$$

$C = \cos(\sigma\sqrt{\lambda}\, t)$ or $\sin(\sigma\sqrt{\lambda}\, t)$

[redacted]

$$\frac{C''}{\sigma^2 C} = \underbrace{\frac{(rA')'}{rA} + \frac{B''}{r^2 B}} = -\lambda$$

$2^{\underline{nd}}$ separation of variables

$$\frac{r(rA')'}{A} + \lambda r^2 = -\frac{B''}{B} = \mu$$

periodic boundary conditions $B = B(\theta)$

$$B_j = a_j \cos(j\theta) + b_j \sin j\theta$$

$$\mu_j = j^2 \quad , \quad j = \underline{\underline{0}}, 1, 2, 3, \ldots$$

[redacted]

$$\boxed{r(rA')' + (\lambda r^2 - j^2)A = 0} \quad \text{for } A = A(r)$$

$$\boxed{A(a) = 0, \quad A(0) \in \mathbb{R}}$$

↑ outer radius

$$r^2 A'' + rA' + (\lambda r^2 - j^2)A = 0$$

↑ singular at r=0

ODE has a singular point at r=0

[redacted]

$$r^2 A'' + r A' + (\lambda r^2 - j^2) A = 0$$

$$A'' + \frac{1}{r} A' + \left(\lambda - \frac{j^2}{r^2}\right) A = 0$$

<u>IDEA</u>: GET RID OF λ

$$r = \frac{\xi}{\sqrt{\lambda}}, \quad y(\xi) = A\left(\frac{\xi}{\sqrt{\lambda}}\right)$$

$$y' = \frac{1}{\sqrt{\lambda}} A', \quad y'' = \frac{1}{\lambda} A''$$

$$\lambda y'' + \frac{\sqrt{\lambda}}{\xi} \cdot \sqrt{\lambda}\, y' + \lambda\left(1 - \frac{j^2}{\xi^2}\right) y = 0$$

$$y'' + \frac{1}{\xi} y' + \left(1 - \frac{j^2}{\xi^2}\right) y = 0$$

Bessel ODE

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Bessel's Equation

$$y'' + \frac{1}{\xi} y' + \left(1 - \frac{\nu^2}{\xi^2}\right) y = 0$$

order ν (maybe not an integer)

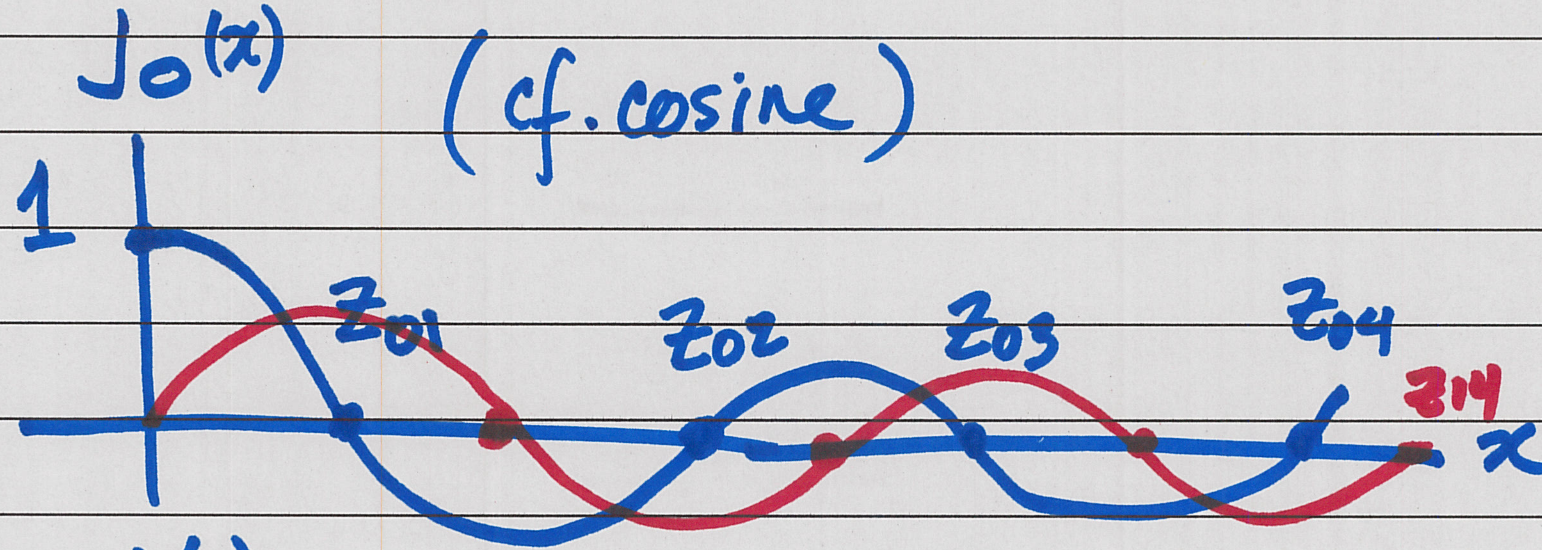
For us $\nu = j = 0, 1, 2, 3, \ldots$

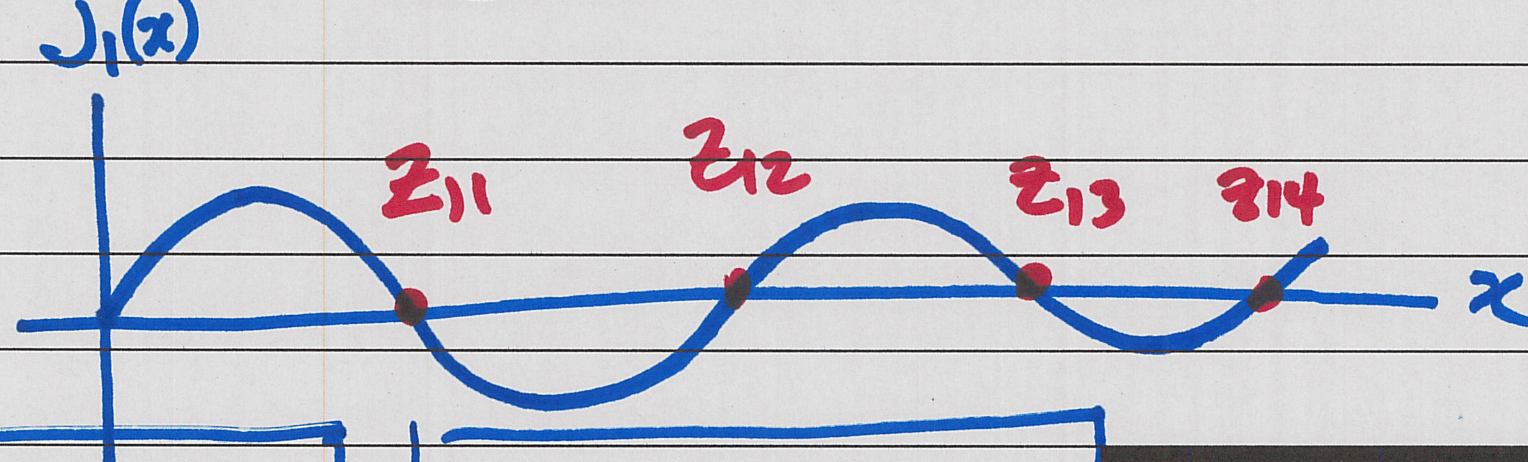
$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2) y = 0}$$

Basis of solutions

J_ν, Y_ν "first and second kinds"

(cf. cosine)





$$\boxed{J_0' = -J_1}$$

$$\boxed{J_1' = \frac{1}{2}\left[J_0 + J_2\right]}$$

[redacted]

$$\boxed{A_j\Big(\underbrace{\frac{\xi}{\sqrt{\lambda}}}_{r}\Big) = c_j\, J_\nu(\xi)}\ , \quad \nu = j = 0,1,2,\ldots$$

$A_j(0) \in \mathbb{R} \rightarrow$ no Y_ν

$$A_j(r) = c_j\, J_\nu\left(\sqrt{\lambda}\, r\right) \ , \quad A_j(a) = 0$$

$$\sqrt{\lambda}\, a = Z_{\nu k}$$

$$A_{jk}(r) = c_{jk}\, J_\nu\left(Z_{jk}\,\frac{r}{a}\right)$$

$$\nu = j = 0,1,2,\ldots$$

[redacted]

Fundamental Modes of The circular drum

$$\boxed{C_{jk}\left(a_j \cos j\theta + b_j \sin j\theta\right) J_\nu\!\left(z_{jk}\frac{r}{a}\right)}$$

$\nu = j = 0, 1, 2, \ldots$
$k = 1, 2, 3, \ldots$

oscillation

$$\alpha_{jk}\cos\!\left(\sigma\frac{z_{jk}}{a}t\right) + \beta_{jk}\sin\!\left(\sigma\frac{z_{jk}}{a}t\right)$$

$$\boxed{J_\nu\left(z_{jk}\right) = 0}$$

[redacted]