MATH 4581 Lecture 6 Thursday Sept. 9, 2021

Last time:

$$\begin{cases} u_t = k\, u_{xx} \quad \text{on } (0,L)\times(0,T) \\ u_x(0,t) = 0 = u_x(L,t), \quad 0<t<T \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = \underbrace{u_0}_{\text{const.}} + \sum_{j=1}^{\infty} a_j\, e^{-k\frac{j^2\pi^2}{L^2}t} \cos\frac{j\pi}{L}x$$

$$u_0 = \frac{1}{L}\int_0^L g(x)\,dx, \quad a_j = \frac{2}{L}\int_0^L g(x)\cos\frac{j\pi}{L}x\,dx$$ [redacted]

Transport / Convection

transporting mass:

Start with a vector (velocity) field:

$$\mathbb{V} \quad , \quad [\mathbb{V}] = \frac{L}{T}$$

volumetric mass density ρ , $[\rho] = \frac{M}{L^3}$

$$[\rho \mathbb{V}] = \frac{M}{T L^2}$$

These are dimensions of a flux

[redacted]

rate at which mass is <u>transported</u> across a surface S is

$$\int_S \rho \mathbb{v} \cdot n$$

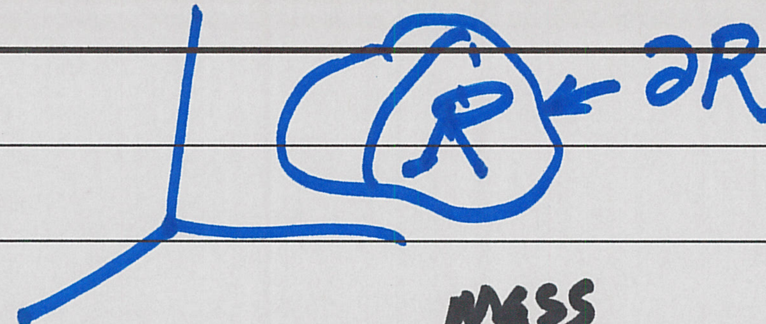
In <u>particular</u>,

$$\int_{\partial R} \rho \mathbb{v} \cdot n$$

rate of mass going <u>out</u> of R

<u>unit</u> outward normal

[redacted]



Mass Conservation:

$\rho = \rho(x, t)$ spatially dependent mass density

$$\underbrace{\frac{d}{dt}\int_R \rho}_{\text{rate at which mass enters } R} = -\int_{\partial R} \rho \mathbb{V}\cdot\vec{n} = -\int_R \operatorname{div}(\rho \mathbb{V})$$

$$\boxed{\rho_t = -\operatorname{div}(\rho \mathbb{V})}$$

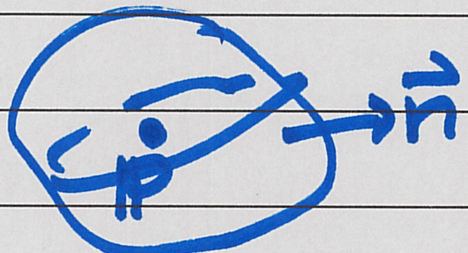
Continuity Equation.

[redacted]

What is the divergence of a vector field?

BAD ANSWER $\operatorname{div} \mathbb{V} = \frac{\partial v_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}$

GOOD ANSWER LOOK AT $\int_{\partial R} \mathbb{V} \cdot \vec{n}$



imagine $\lim_{R \to \{P\}} \int_{\partial R} \mathbb{V} \cdot \vec{n}$

[redacted]

$$\lim_{R \to \{\mathbb{P}\}} \int_{\partial R} \mathbb{V} \cdot \vec{n} = 0$$

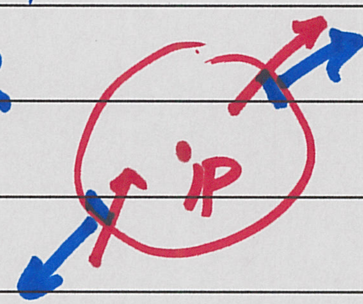
$$\sum \underbrace{\mathbb{V}(\vec{x}_j) \cdot \vec{n}(\vec{x}_j)}_{\leq |\mathbb{V}|} A_j$$

A_j ← little areas

$\sum_i A_j = \text{Area}(\partial R)$

TRY $$\lim_{R \to \{\mathbb{P}\}} \frac{1}{\mu(\partial R)} \int_{\partial R} \mathbb{V} \cdot \vec{n} = 0$$

$\mu(\partial R)$ ← area of ∂R



$\sim \frac{0}{0}$

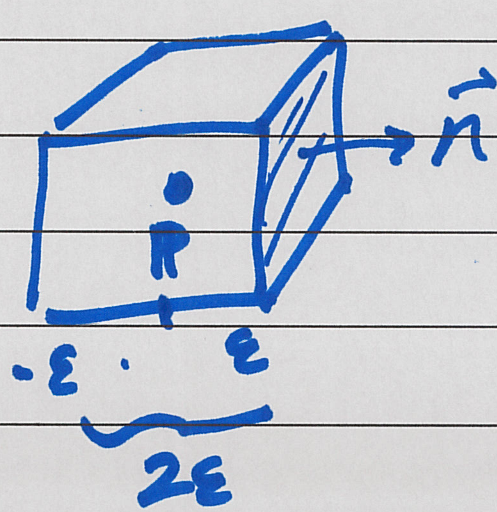
[redacted]

Good Answer

$$\boxed{\operatorname{div} \mathbb{V}(\mathbb{P}) = \lim_{R \to \{\mathbb{P}\}} \frac{1}{\mu(R)} \int_{\partial R} \mathbb{V} \cdot \vec{n}}$$

volume (n-dim'l measure) of R

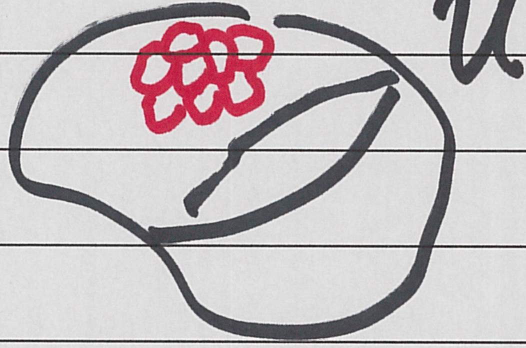
Exercise:



[redacted]

$$\operatorname{div} \mathbb{V}(P) = \lim_{R \to \{P\}} \frac{1}{\operatorname{vol}(R)} \int_{\partial R} \mathbb{V} \cdot \vec{n}$$

"PROOF" of The DIVERGENCE THM:



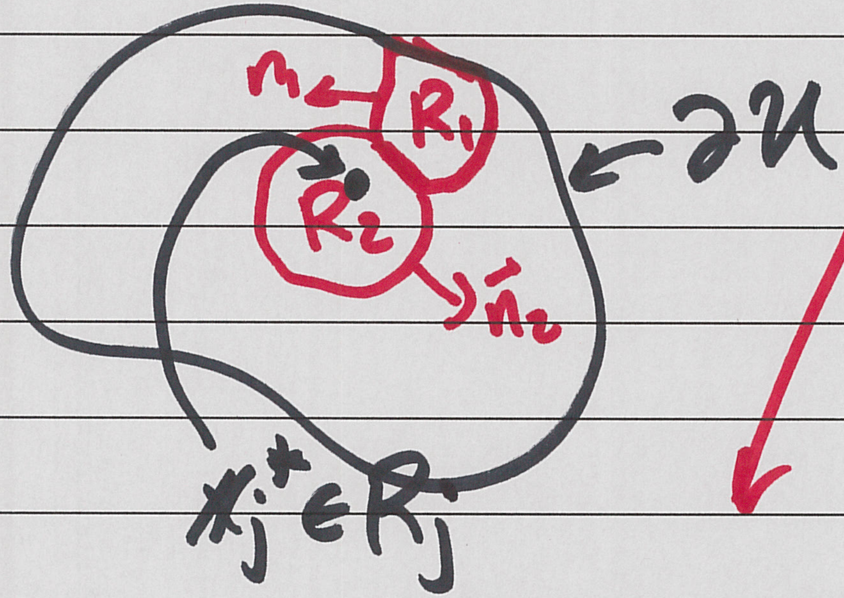
$$\int_{\mathcal{U}} \operatorname{div} \mathbb{V} = \int_{\partial \mathcal{U}} \mathbb{V} \cdot \vec{n}$$

cut $\mathcal{U}$ up into little regions

R_j (not overlapping - nice)

[redacted]

$$\int_{\partial \mathcal{U}} \mathbb{V}\cdot\vec{n} = \sum_j \int_{\partial R_j} \mathbb{V}\cdot n_j \qquad (?)$$



$$\operatorname{div} \mathbb{V}(x_j^*) \sim \frac{1}{\operatorname{vol}(R_j)} \int_{\partial R_j} \mathbb{V}\cdot\vec{n}_j$$

$$\sim \sum_j \operatorname{div} \mathbb{V}(x_j^*) \operatorname{vol}(R_j)$$

$$\to \int_{\mathcal{U}} \operatorname{div} \mathbb{V}$$

[redacted]